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Bogoliubov method in description of nuclear rotation

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Abstract

The problem of identifying and extracting the dynamic variables associated with symmetry transformations from the full set of dynamic variables is considered. It is demonstrated that employing a boson representation of bifermion operators enables the problem to be solved using the canonical transformation of dynamic variables proposed by N. N. Bogoliubov. The results obtained justify the application of the cranking model for the description of the rotational excitations of nuclei.

Keywords: intrinsic coordinate system, boson representation, cranking model

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1. Introduction

The microscopic Hamiltonian used for description of the rotational excited states of nuclei is usually written in the intrinsic coordinate system, i.e., in the coordinate system rotating with the average field of the nucleus. In order to justify this Hamiltonian, it is crucial to develop a mathematical technique to transform the Hamiltonian given in the laboratory frame to the Hamiltonian given in the intrinsic frame.

The issue of transition to the intrinsic coordinate system is closely related to the difficulties that arise when allocating a self-consistent nuclear mean field within microscopic models. The fact is that the methods used in this procedure (Hartree–Fock method, Bogoliubov u–v transformation) violate the laws of the conservation of space transition momentum, rotational momentum, and the number of particles. It is because the dynamic variables that are parameters of the symmetry group of the Hamiltonian are not separated from the other dynamic variables. Therefore, after applying approximate methods the Hamiltonian loses invariance with respect to transformations belonging to the group of its symmetry. If it were possible to isolate the variables that are parameters of the symmetry group of the Hamiltonian, and only after that to apply the approximate methods to the Hamiltonian, which depends on the remaining variables, then there would be no violation of the conservation laws. In fact, the

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selection of variables, which are parameters of the symmetry group, is essentially a transition to the intrinsic coordinate system.

There were several attempts to develop a unified theoretical description of both vibrational and rotational states of nuclei in spite of the fact that nuclear rotation is not characterized by small oscillations of the collective degrees of freedom. In this respect it is necessary to mention the series of works by E. R. Marshalek et al. [1–3] where the boson representation of fermion operators proposed by S. T. Belyaev and V. G. Zelevinsky [4] was taken as a basis. The paper by A. Kerman and A. Klein [5] should also be noted where a method was formulated for describing rotation by some generalization of the Random Phase Approximation (RPA) method. A completely different approach was suggested in the paper by S. T. Belyaev and V. G. Zelevinsky [6] to isolate dynamic variables describing rotational motion.

The present work is aimed at deriving with the help of the Bogoliubov method the cranking model Hamiltonian from the Hamiltonian presented in the general form. This method clearly demonstrates the conservation of the angular momentum and the separation of non-physical modes, as a fee for the explicit isolation of dynamic variables associated with the symmetry properties of the Hamiltonian.

2. Transition to intrinsic frame

The problem of identifying dynamic variables associated with a symmetry group of the Hamiltonian was principally solved in the work of N. N. Bogoliubov [7] (see also [8, 9]). The Bogoliubov method involves a canonical transformation of variables, resulting in the introduction of parameters from the Hamiltonian's symmetry group as the new variables. Then the question of any transformation of the remaining variables is no longer related to the transformation properties of the Hamiltonian.

Unfortunately, the Bogoliubov method cannot be applied directly to a fermion system whose Hamiltonian is written in the second quantized form. (The problem of selecting the parameters of the symmetry group in the case of the Hamiltonian formulated in terms of coordinates and nucleon momentum was considered in [10–14].)

However, the problem can be solved by using the finite boson representation of bifermion operators. This representation was proposed and discussed in detail in [15] (see also [17]), where it was shown that for bifermion operators the following boson representation is true. It satisfies exactly all commutation relations:

$$
\begin{gathered}
a^+_{sm} a_{s'm'} \to \sum_{tn} b^+_{sm,tn} b_{s'm',tn}, \\
a^+_{sm} a^+_{s'm'} \to b^+_{sm,s'm'} - \sum_{tn,t'n'} b^+_{sm,tn} b^+_{s'm',t'n'} b_{tn,t'n'}, \\
a_{s'm'} a_{sm} \to b_{sm,s'm'},
\end{gathered}
\tag{1}
$$

where $a^+_{sm}(a_{sm})$ is the fermion creation (annihilation) operator. The boson creation (annihilation) operator $b^+_{sm,s'm'}(b_{sm,s'm'})$ satisfies the following commutation relation:

$$
\begin{gathered}
[b_{sm,tn}, b^+_{s'm',t'n'}] = \delta_{sm,s'm'}\delta_{tn,t'n'} - \delta_{sm,t'n'}\delta_{s'm',tn}, \\
b_{sm,s'm'} = -b_{s'm',sm},
\end{gathered}
$$

where $s \equiv nlj$ is a set of quantum numbers which characterize a single particle state, m is a projection of the single particle angular momentum on the z axis of the laboratory coordinate system (see a review paper [16]).

It is convenient to introduce boson operators with well-defined angular momentum and its projection:

$$b_{sm,tn} = \sqrt{2}\sum_{\lambda\mu} C^{\lambda\mu}_{j_s m j_t n} b_{\lambda\mu}(st). \tag{2}$$

For boson operators, the following representation in terms of generalized coordinates and conjugate momenta is known:

$$\begin{aligned} b^+_{\lambda\mu}(st) &\to f_{\lambda\mu}(st),\\ b_{\lambda\mu}(st) &\to \frac{\partial}{\partial f_{\lambda\mu}(st)}. \end{aligned} \tag{3}$$

Using this representation, we can express the Hamiltonian in terms of generalized coordinates and momenta and to take advantage of the idea of Bogoliubov transformation. As an example, we present the expressions for the particle number $\hat{N}$ and angular momentum operators $\hat{I}_\mu$ in terms of the new dynamic variables:

$$\hat{N} = \sum_{\lambda\mu st} 2f_{\lambda\mu}(st)\frac{\partial}{\partial f_{\lambda\mu}(st)}, \tag{4}$$

$$\hat{I}_\mu = \sum_{\lambda\eta\eta' st} \sqrt{\lambda(\lambda+1)} C^{\lambda\eta}_{\lambda\eta' 1\mu} f_{\lambda\eta}(st)\frac{\partial}{\partial f_{\lambda\eta'}(st)}. \tag{5}$$

Note that the operators $\hat{N}$ and $\hat{I}_\mu$ depend on all dynamic variables $f_{\lambda\mu}(st)$.

For simplicity, we consider below a model Hamiltonian with pairing and quadrupole residual forces only, although consideration can be done with the Hamiltonian of a more general form:

$$\begin{gathered} H = H_0 + H_{\text{pair}} + H_{QQ}, \\ H_0 = \sum_s \varepsilon_s N_s, \\ H_{pair} = -\frac{G}{4}A^+A, \\ H_{QQ} = -\kappa\sum_\mu (-1)^\mu Q_{2\mu}Q_{2-\mu}, \\ N_s = \sum_m a^+_{sm}a_{sm}, \\ A^+ = \sum_{sm}(-1)^{j_s-m}a^+_{sm}a^+_{s-m}, \\ A = (A^+)^+, \\ Q_{2\mu} = \sum_{ss'mm'} \langle sm|r^2Y_{2\mu}|s'm'\rangle a^+_{sm}a_{s'm'}. \end{gathered} \tag{6}$$

In terms of the operators $f_{\lambda\mu}(st)$, $\frac{\partial}{\partial f_{\lambda\mu}(st)}$,

$$\begin{aligned}
N_s &= \sum_{\lambda\mu t} 2 f_{\lambda\mu}(st)\frac{\partial}{\partial f_{\lambda\mu}(st)},\\
A^+ &= \sum_s 2\sqrt{j_s+1/2}f_{00}(ss) - 2\sqrt{2}\sum_{ss't\lambda\lambda'\lambda''\mu\mu'\mu''}\sqrt{(2\lambda+1)(2\lambda'+1)}\times\\
&\times(-1)^{j_s+j_t+\lambda+\lambda''}\left\{\begin{array}{ccc}\lambda'' & j_{s'} & j_t\\ j_s & \lambda' & \lambda\end{array}\right\}C^{\lambda''\mu''}_{\lambda\mu\lambda'\mu'}f_{\lambda\mu}(ss')f_{\lambda'\mu'}(st)\frac{\partial}{\partial f_{\lambda''\mu''}(s't)},\\
A &= \sum_s 2\sqrt{j_s+1/2}\frac{\partial}{\partial f_{00}(ss)},\\
Q_{2\mu} &= 2\sum_{ss't\lambda\lambda'\eta\eta'}(-1)^{j_s+j_t-\lambda'}\sqrt{2\lambda'+1}\langle s||r^2Y_2||s'\rangle\left\{\begin{array}{ccc}j_t & j_{s'} & \lambda'\\ 2 & \lambda & j_s\end{array}\right\}\times\\
&\times C^{\lambda\eta}_{\lambda'\eta'2\mu}f_{\lambda\eta}(st)\frac{\partial}{\partial f_{\lambda'\eta'}(s't)}.
\end{aligned}\tag{7}$$

Static pair correlations and quadrupole deformation can occur in the system described by this Hamiltonian. Commonly used approximate methods that include these effects lead to the loss of Hamiltonian invariance under rotations in phase space (linked to particle number conservation) and in three-dimensional space (linked to angular momentum conservation). Therefore, from dynamic variables describing nucleus it is necessary from the very beginning to separate the angle ϕ, which is canonically conjugate to the operator of the number of particles $\hat{N} = -2\imath\frac{\partial}{\partial\phi}$, and the Euler angles θ_l, in terms of which the angular momentum operator is expressed. Through the separation of these variables, we move into the intrinsic coordinate system while preserving the conservation laws for particle number and angular momentum.

Let us demonstrate that the problem of separating the dynamical variables associated with the Hamiltonian's symmetry can be resolved by applying the following transformation of the dynamic variable:

$$f_{\lambda\mu}(st) = \exp(\imath\phi)\sum_k D^{\lambda}_{\mu k}(\vec{\theta})F_{\lambda k}(st),\tag{8}$$

where $D^{\lambda}_{\mu k}(\vec{\theta})$ is the Wigner function. Since there are four more new variables than the old ones, it is necessary to impose four additional conditions on $F_{\lambda k}(st)$:

$$\begin{aligned}\Psi\left(F_{\lambda k}(st)\right) &= 0,\\ \Phi_\eta\left(F_{\lambda k}(st)\right) &= 0,\end{aligned}\tag{9}$$

where η takes three different values. The specific form of the additional conditions will be discussed below.

Let us find the expression for $\frac{\partial}{\partial f_{\lambda\mu}(st)}$ in terms of the new variables:

$$\frac{\partial}{\partial f_{\lambda\mu}(st)} = \frac{\partial\phi}{\partial f_{\lambda\mu}(st)}\frac{\partial}{\partial\phi} + \sum_{l=1}^{3}\frac{\partial\theta_l}{\partial f_{\lambda\mu}(st)}\frac{\partial}{\partial\theta_l} + \sum_{\lambda'k's't'}\frac{\partial F_{\lambda'k'}(s't')}{\partial f_{\lambda\mu}(st)}\frac{\partial}{\partial F_{\lambda'k'}(s't')}.\tag{10}$$

The quantity $\frac{\partial F_{\lambda'k'}(s't')}{\partial f_{\lambda\mu}(st)}$ can be found using the relation inverse to (8):

$$F_{\lambda k}(st) = \exp(-\imath\phi)\sum_\mu (D^{\lambda}_{\mu k})^* f_{\lambda\mu}(st)\tag{11}$$

and the Wigner function differentiation rule [18] which looks like

$$\frac{\partial D^{\lambda}_{\mu k}}{\partial \theta_l} = \imath \sum_{k'\eta} D^{\lambda}_{\mu k'} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta} q_{\eta l}(\vec{\theta}), \tag{12}$$

where $q_{\eta l}(\vec{\theta})$ is the rotational matrix [18]. Then

$$\begin{aligned} \frac{\partial F_{\lambda' k'}(s't')}{\partial f_{\lambda\mu}(st)} = \exp(-\imath\phi)(D^{\lambda}_{\mu k'})^* \delta_{\lambda\lambda'}\delta_{st,s't'} - \imath \frac{\partial \phi}{\partial f_{\lambda\mu}(st)} F_{\lambda' k'}(s't') - \\ -\imath \sum_{k''\eta l} \frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)} \sqrt{\lambda'(\lambda'+1)} C^{\lambda' k''}_{\lambda' k' 1\eta} q_{\eta l} F_{\lambda' k''}(s't'). \end{aligned} \tag{13}$$

As a result, for $\frac{\partial}{\partial f_{\lambda\mu}(st)}$ we obtain

$$\begin{aligned} \frac{\partial}{\partial f_{\lambda\mu}(st)} = \exp(-\imath\phi) \sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\partial}{\partial F_{\lambda k'}(st)} + \imath \frac{\partial \phi}{\partial f_{\lambda\mu}(st)} \left(-\imath \frac{\partial}{\partial \phi} - \frac{\hat{n}}{2}\right) + \\ + \sum_l \frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)} \left(\frac{\partial}{\partial \theta_l} - \imath \sum_{\eta} q_{\eta l}(\vec{\theta}) \hat{L}_{\eta}\right), \end{aligned} \tag{14}$$

where

$$\hat{n} = 2 \sum_{\lambda k s t} F_{\lambda k}(st) \frac{\partial}{\partial F_{\lambda k}(st)}, \tag{15}$$

$$\hat{L}_{\eta} = \sum_{\lambda k k'} \sum_{st} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta} F_{\lambda k}(st) \frac{\partial}{\partial F_{\lambda k'}(st)}. \tag{16}$$

If we use the well-known expression for the angular momentum projection operator on the axis of the intrinsic coordinate system:

$$\hat{\mathcal{L}}_{\eta} = -\imath \sum_l q^{-1}_{\eta l} \frac{\partial}{\partial \theta_l}, \tag{17}$$

then

$$\begin{aligned} \frac{\partial}{\partial f_{\lambda\mu}(st)} = \imath \frac{\partial \phi}{\partial f_{\lambda\mu}(st)} \left(-\imath \frac{\partial}{\partial \phi} - \frac{\hat{n}}{2}\right) + \imath \sum_{l\eta} \frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)} q_{\eta l} \left(\hat{\mathcal{L}}_{\eta} - \hat{L}_{\eta}\right) + \\ + \exp(-\imath\phi) \sum_{k'} \left(D^{\lambda}_{\mu k'}\right)^* \frac{\partial}{\partial F_{\lambda k'}(st)}. \end{aligned} \tag{18}$$

In order to find the expressions for $\frac{\partial \phi}{\partial f_{\lambda\mu}(st)}$ and $\frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)}$, it is necessary to use the additional conditions (9) from which it follows that

$$\frac{\partial \Phi_{\eta}}{\partial f_{\lambda\mu}(st)} = \sum_{\lambda' k' s' t'} \frac{\partial F_{\lambda' k'}(s't')}{\partial f_{\lambda\mu}(st)} \frac{\partial \Phi_{\eta}}{\partial F_{\lambda' k'}(s't')} = 0 \tag{19}$$

and the analogous relation for Ψ is

$$\frac{\partial \Psi}{\partial f_{\lambda\mu}(st)} = \sum_{\lambda' k' s' t'} \frac{\partial F_{\lambda' k'}(s't')}{\partial f_{\lambda\mu}(st)} \frac{\partial \Psi}{\partial F_{\lambda' k'}(s't')} = 0. \tag{20}$$

Applying Eq. (14) to Φ_η, we obtain

$$-\frac{\imath}{2}\frac{\partial \phi}{\partial f_{\lambda\mu}(st)}[\hat{n},\Phi_\eta] + \sum_l (-\imath) \sum_{\eta'} q_{\eta' l}[\hat{L}_{\eta'},\Phi_\eta]\frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)} + \\ + \exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\partial \Phi_\eta}{\partial F_{\lambda k'}(st)} = 0 \tag{21}$$

and analogous relation including Ψ

$$-\frac{\imath}{2}\frac{\partial \phi}{\partial f_{\lambda\mu}(st)}[\hat{n},\Psi] - \imath \sum_l \sum_{\eta'} q_{\eta' l}[\hat{L}_{\eta'},\Psi]\frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)} + \\ + \exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\partial \Psi}{\partial F_{\lambda k'}(st)} = 0. \tag{22}$$

From the last two relations, we get

$$-\frac{\imath}{2}\frac{\partial \phi}{\partial f_{\lambda\mu}(st)}[\hat{n},\Phi_\eta] + \exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\partial \Phi_\eta}{\partial F_{\lambda k'}(st)} = \\ = \imath \sum_{l\eta'} q_{\eta' l}\frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)}[\hat{L}_{\eta'},\Phi_\eta] \tag{23}$$

and

$$-\frac{\imath}{2}\frac{\partial \phi}{\partial f_{\lambda\mu}(st)}[\hat{n},\Psi] + \exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\partial \Psi}{\partial F_{\lambda k'}(st)} = \\ = \imath \sum_{l\eta'} q_{\eta' l}\frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)}[\hat{L}_{\eta'},\Psi]. \tag{24}$$

As the functions Ψ and Φ_η, we take the eigenfunctions of the operator $\hat{n}$. This means that Ψ and Φ_η are polynomials of degree n. Then $[\hat{n},\Phi_\eta] = n\Phi_\eta$ and $[\hat{n},\Psi] = n\Psi$.

Suppose Ψ is a scalar with respect to the vector operator $\hat{L}_\eta$, i.e., $[\hat{L}_\eta,\Psi] = 0$. Then from (24) it follows that

$$\imath\frac{\partial \phi}{\partial f_{\lambda\mu}(st)} = 2\exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{1}{n\Psi}\frac{\partial \Psi}{\partial F_{\lambda k'}(st)}. \tag{25}$$

Substituting this result into (23) and (18), we obtain

$$\imath\sum_l \frac{\partial \theta_l}{\partial f_{\lambda\mu}(st)} \sum_{\eta'} q_{\eta' l}(\vec{\theta})[\hat{L}_{\eta'},\Phi_\eta] = -\exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\Phi_\eta}{\Psi}\frac{\partial \Psi}{\partial F_{\lambda k'}(st)} + \\ + \exp(-\imath\phi)\sum_{k'} (D^{\lambda}_{\mu k'})^* \frac{\partial \Phi_\eta}{\partial F_{\lambda k'}(st)} \tag{26}$$

and

$$\frac{\partial}{\partial f_{\lambda\mu}(st)} = \exp\left(-\imath\phi\right)\sum_{k'}(D^{\lambda}_{\mu k'})^*\left(\frac{1}{n\Psi}\frac{\partial\Psi}{\partial F_{\lambda k'}(st)}(\hat{N}-\hat{n})+\frac{\partial}{\partial F_{\lambda k'}(st)}\right)+ \\ +\imath\sum_{l}\frac{\partial\theta_l}{\partial f_{\lambda\mu}(st)}\sum_{\eta}q_{\eta l}(\vec{\theta})(\hat{\mathcal{L}}_{\eta}-\hat{L}_{\eta}). \tag{27}$$

Let us introduce a new notation

$$\imath\sum_{l}\frac{\partial\theta_l}{\partial f_{\lambda\mu}(st)}q_{\eta l}(\vec{\theta}) = \exp\left(-\imath\phi\right)\sum_{k'}(D^{\lambda}_{\mu k'})^*\cdot A_{\lambda k',-\eta}(-1)^{1-\eta}. \tag{28}$$

Then Eqs. (26) and (27) can be rewritten as

$$\sum_{\eta'}(-1)^{1-\eta'}A_{\lambda k',-\eta'}(st)\cdot[\hat{L}_{\eta'},\Phi_{\eta}] = -\frac{\Phi_{\eta}}{\Psi}\frac{\partial\Psi}{\partial F_{\lambda k'}(st)}+\frac{\partial\Phi_{\eta}}{\partial F_{\lambda k'}(st)} \tag{29}$$

and

$$\frac{\partial}{\partial f_{\lambda\mu}(st)} = \exp\left(-\imath\phi\right)\sum_{k'}(D^{\lambda}_{\mu k'})^*\Bigg(\frac{1}{n\Psi}\frac{\partial\Psi}{\partial F_{\lambda k'}(st)}(\hat{N}-\hat{n})+\frac{\partial}{\partial F_{\lambda k'}(st)}+ \\ +\sum_{\eta}(-1)^{1-\eta}A_{\lambda k',-\eta}(st)(\hat{\mathcal{L}}_{\eta}-\hat{L}_{\eta})\Bigg). \tag{30}$$

Until now, the function Φ_η has been treated as arbitrary, without assuming that Φ_η is an n-degree polynomial. In what follows, we will take Φ_η to represent the components of an arbitrary vector that transforms under the action of the operator $\hat{L}$ as follows:

$$[\hat{L}_{\eta'},\Phi_{\eta}] = -\sqrt{2}C^{1\lambda}_{1\eta'1\eta}\Phi_{\lambda}. \tag{31}$$

Using this relation, we obtain

$$\sum_{\eta}(-1)^{1-\eta'}A_{\lambda k',-\eta'}(st)\cdot[\hat{L}_{\eta'},\Phi_{\eta}] = \sqrt{2}\sum_{\eta'\lambda}C^{1\eta}_{1\eta'1\lambda}A_{\lambda k',\eta'}(st)\Phi_{\lambda}. \tag{32}$$

Substituting (32) into (29), we get

$$\sqrt{2}\sum_{\eta'\lambda}C^{1\eta}_{1\eta'1\lambda}A_{\lambda k',\eta'}(st)\Phi_{\lambda} = -\frac{\Phi_{\eta}}{\Psi}\frac{\partial\Psi}{\partial F_{\lambda k'}(st)}+\frac{\partial\Phi_{\eta}}{\partial F_{\lambda k'}(st)}. \tag{33}$$

The expression on the left-hand side of (33) is the vector product of $\vec{A}_{\lambda k'}(st)$ and $\vec{\Phi}$. In the general case, knowledge of the vector product does not allow us to uniquely determine one of the vectors in the vector product if the other is known. This means that in the general case we cannot determine the vector $\vec{A}_{\lambda k'}(st)$ from the relation (33). This ambiguity can be eliminated if we search for the solution only among vectors orthogonal to $\vec{\Phi}$. Then

$$A_{\lambda k',\eta}(st) = \frac{\sqrt{2}}{\sum_{\mu}\Phi_{\mu}\Phi_{-\mu}}\sum_{\mu\nu}C^{1\eta}_{1\mu 1\nu}\Phi_{\mu}\left(\frac{\partial\Phi_{\nu}}{\partial F_{\lambda k'}(st)}-\frac{\Phi_{\nu}}{\Psi}\frac{\partial\Psi}{\partial F_{\lambda k'}(st)}\right). \tag{34}$$

Substituting (34) in (30), we obtain

$$\frac{\partial}{\partial f_{\lambda\mu}(st)} = \exp\left(-\imath\phi\right) \sum_{k'} (D^{\lambda}_{\mu k'})^{*} \left(\frac{\partial}{\partial F_{\lambda k'}(st)} + \frac{1}{n\Psi} \frac{\partial \Psi}{\partial F_{\lambda k'}(st)} (\hat{N} - \hat{n}) + \right.$$
$$\left. + \sum_{\eta} (-1)^{1-\eta} \frac{\sqrt{2}}{\sum_{\nu} (-1)^{\nu} \Phi_{\nu} \Phi_{-\nu}} \sum_{\nu\lambda} C^{1\eta}_{1\nu 1\lambda} \Phi_{\nu} \left(\frac{\partial \Phi_{\lambda}}{\partial F_{\lambda k'}(st)} - \frac{\Phi_{\lambda}}{\Psi} \frac{\partial \Psi}{\partial F_{\lambda k'}(st)} \right) \cdot (\hat{\mathcal{L}}_{\eta} - \hat{L}_{\eta}) \right). \quad (35)$$

It is easy to check that $[\frac{\partial}{\partial f_{\lambda\mu}(st)}, \Phi_{\eta}] = [\frac{\partial}{\partial f_{\lambda\mu}(st)}, \Psi] = 0$. This means that additional conditions (9) are compatible with the Hamiltonian. Substituting (35) into (4) and (5), we obtain

$$\begin{aligned} \hat{N} &= -2\imath \frac{\partial}{\partial \phi}, \\ \hat{I}_{\mu} &= \sum_{\eta} D^{1}_{\mu\eta}(\vec{\theta}) \hat{\mathcal{L}}_{\eta}. \end{aligned} \quad (36)$$

In the new variables, the Hamiltonian takes the form

$$H = \tilde{H}_0 - \frac{G}{4} \tilde{A}^{+} \tilde{A} - \kappa \sum_{k} (-1)^{k} Q_{2k} Q_{2-k}, \quad (37)$$

where

$$\tilde{H}_0 = 2 \sum_{\lambda k s t} \varepsilon_s F_{\lambda k}(st) \frac{\partial}{\partial g_{\lambda k}(st)},$$
$$\tilde{A} = 2 \sum_{s} \sqrt{j_s + 1/2} \frac{\partial}{\partial g_{00}(ss)},$$
$$\tilde{A}^{+} = 2 \sum_{s} \sqrt{j_s + 1/2} F_{00}(ss) - 2\sqrt{2} \sum_{ss't\lambda\lambda'} \sum_{\lambda'' k k' k''} \sqrt{(2\lambda+1)(2\lambda'+1)} (-1)^{j_s + j_t + \lambda + \lambda''} \times$$
$$\times \left\{ \begin{array}{ccc} \lambda'' & j_{s'} & j_t \\ j_s & \lambda' & \lambda \end{array} \right\} C^{\lambda'' k''}_{\lambda k \lambda' k'} F_{\lambda k}(ss') F_{\lambda' k'}(st) \frac{\partial}{\partial g_{\lambda'' k''}(s't)},$$
$$Q_{2k} = 2 \sum_{ss't\lambda\lambda'} (-1)^{j_s + j_t - \lambda'} \sqrt{2\lambda'+1} \langle s || r^2 Y_2 || s' \rangle \times \left\{ \begin{array}{ccc} j_t & j_{s'} & \lambda' \\ 2 & \lambda & j_s \end{array} \right\} C^{\lambda k'}_{\lambda' k'' 2k} F_{\lambda k'}(st) \frac{\partial}{\partial g_{\lambda' k''}(s't)}.$$

Here we use the notation

$$\frac{\partial}{\partial g_{\lambda k}(st)} \equiv \frac{\partial}{\partial F_{\lambda k}(st)} + \frac{1}{n\Psi} \frac{\partial \Psi}{\partial F_{\lambda k}(st)} (\hat{N} - \hat{n}) +$$
$$+ \sum_{\eta} (-1)^{1-\eta} \frac{\sqrt{2}}{\sum_{\nu} (-1)^{\nu} \Phi_{\nu} \Phi_{-\nu}} \sum_{\nu\lambda} C^{1\eta}_{1\nu 1\lambda} \Phi_{\nu} \left(\frac{\partial \Phi_{\lambda}}{\partial F_{\lambda k}(st)} - \frac{\Phi_{\lambda}}{\Psi} \frac{\partial \Psi}{\partial F_{\lambda k}(st)} \right) \times (\hat{\mathcal{L}}_{\eta} - \hat{L}_{\eta}). \quad (38)$$

For the consideration below it is important to note that, due to our choice of functions Ψ and Φ_{η}, the Hamiltonian (37) commutes with the operators $\hat{n}$ and $\hat{L}_{\eta}$:

$$[H, \hat{n}] = 0, \quad (39)$$
$$[H, \hat{L}_{\eta}] = 0. \quad (40)$$

The Hamiltonian (37) depends on ϕ and θ_l through the operators $\hat{N}$ and $\hat{\mathcal{L}}_\eta$ only. The absence of an explicit dependence on θ_l and ϕ means that the Hamiltonian is invariant under rotation in three-dimensional and phase spaces, and eigenvalues of $\hat{N}$ and $\hat{\vec{\mathcal{L}}}^2$ are conserved. It should be emphasized that we obtained a finite-term expression for the Hamiltonian, rather than an infinite series, due to the use of the finite boson representation.

Let us show that the expression (37) for the Hamiltonian can be simplified. Since the operator of the number of particles $\hat{N}$ commutes with the Hamiltonian, it can be replaced by its eigenvalue N. As the consequence, the operator $\hat{n}$ in (39) can also be replaced by its eigenvalue, which we choose to be N. Thus, the operator $(\hat{N} - \hat{n})$ gives zero acting on any eigenfunction of the Hamiltonian. This result, however, introduces an additional condition

$$2\sum_{\lambda kst} F_{\lambda k}(st)\frac{\partial}{\partial F_{\lambda k}(st)} = N. \tag{41}$$

Consider the action of the operator $(\hat{\mathcal{L}}_\eta - \hat{L}_\eta)$ on the eigenfunctions of the Hamiltonian. According to the relation (40) and the rotational invariance of the total Hamiltonian, the eigenfunctions of the Hamiltonian can be presented in the following way:

$$\xi_{IK}(\vec{\theta}, F_{\lambda k}) = \sum_K D^I_{\mu K}(\vec{\theta})U_{IK}(F_{\lambda k}), \tag{42}$$

where the function U_{IK} is constructed so that [19]

$$L_\eta U_{IK}(F_{\lambda k}) = -\sqrt{I(I+1)}C^{IK+\eta}_{1\eta IK}U_{IK+\eta}(F_{\lambda k}). \tag{43}$$

The intrinsic angular momentum operator $\mathcal{L}_\eta$ acting on the Wigner function $D^I_{\mu K}(\vec{\theta})$ gives the following result:

$$\mathcal{L}_\eta D^I_{\mu K} = (-1)^\eta\sqrt{I(I+1)}C^{IK-\eta}_{IK1-\eta}D^I_{\mu K-\eta}. \tag{44}$$

Combining (43) and (44), we can see

$$(\hat{\mathcal{L}}_\eta - \hat{L}_\eta)\xi_{IK} = 0. \tag{45}$$

This means that the action of the operator $\hat{L}_\eta$ on the eigenfunction of the Hamiltonian is equivalent to the action of the operator $\hat{\mathcal{L}}_\eta$. From these results it follows that we can omit in the Hamiltonian the terms proportional to $(\hat{N} - \hat{n})$ and $(\hat{\mathcal{L}}_\eta - \hat{L}_\eta)$.

The following step is to determine a self-consistent nuclear mean field using canonical transformation:

$$\begin{aligned} F_{\lambda k}(st) &\to \Gamma_{\lambda k}(st) + \beta^+_{\lambda k}(st), \\ \frac{\partial}{\partial F_{\lambda k}(st)} &\to \Delta_{\lambda k}(st) + \beta_{\lambda k}(st), \end{aligned} \tag{46}$$

where $\beta^+_{\lambda k}(st)$ ($\beta_{\lambda k}(st)$) is a boson creation (annihilation) operator, $\Gamma_{\lambda k}(st)$ and $\Delta_{\lambda k}(st)$ are c-numbers. Let us replace in the Hamiltonian (37) $F_{\lambda k}(st)$ by $\Gamma_{\lambda k}(st)$ and $\frac{\partial}{\partial F_{\lambda k}(st)}$ by $\Delta_{\lambda k}(st)$. As a result, the expression for the Hamiltonian takes the form of the energy functional obtained by performing Hartree–Fock–Bogoliubov factorization of H which we denote by H_{cc}. This is especially easy to see if we rewrite the expression for H_{cc} in terms of the elements of the density matrix: $\langle a^+_{sm} a_{s'm'}\rangle$, $\langle a_{s'm'} a_{sm}\rangle$, $\langle a^+_{sm} a^+_{s'm'}\rangle$, which can be expressed in terms of $\Gamma_{\lambda k}(ss')$ and $\Delta_{\lambda k}(ss')$ by the relations

$$\begin{aligned}\langle a^+_{sm} a_{s'm'}\rangle &= \sum_{t,m} \Gamma_{sm,tn} \Delta_{s'm',tn} \equiv \rho_{sm,s'm'},\\ \langle a_{s'm'} a_{sm}\rangle &= \Delta_{sm,s'm'} \equiv \kappa_{sm,s'm'},\\ \langle a^+_{sm} a^+_{s'm'}\rangle &= \Gamma_{sm,s'm'} - \sum_{tn,t'n'} \Gamma_{sm,tn}\Gamma_{s'm',t'n'}\Delta_{tn,t'n'} \equiv \kappa^*_{sm,s'm'},\end{aligned} \tag{47}$$

where

$$\begin{aligned}\Gamma_{sm,s'm'} &= \frac{1}{\sqrt{2}} \sum_{\lambda k} C^{\lambda k}_{j_s m j_{s'} m'} \Gamma_{\lambda k}(ss'),\\ \Delta_{sm,s'm'} &= \frac{1}{\sqrt{2}} \sum_{\lambda k} C^{\lambda k}_{j_s m j_{s'} m'} \Delta_{\lambda k}(ss').\end{aligned} \tag{48}$$

Starting from this point, our consideration is practically coincides with that realized by E. R. Marshalek and G. Holzwarth in [21]. The only difference is that in [21] the dynamical variables related to the symmetries of the Hamiltonian (ϕ and $\vec{\theta}$) have not been separated before deriving Hartree–Fock–Bogoliubov energy functional, and the infinite Belyaev–Zelevinsky boson representation has been used instead of the finite Dyson-type boson representation.

It is easy to show directly the validity of the following matrix relations:

$$\begin{aligned}\hat{\rho}\times\hat{\rho} - \hat{\kappa}\times\hat{\kappa}^* = \hat{\rho}, &\quad \hat{\rho}\times\hat{\kappa} = \hat{\kappa}\times\hat{\rho}^*,\\ \hat{\rho}^*\times\hat{\rho}^* - \hat{\kappa}^*\times\hat{\kappa} = \hat{\rho}^*, &\quad \hat{\rho}^*\times\hat{\kappa}^* = \hat{\kappa}^*\times\hat{\rho}.\end{aligned} \tag{49}$$

Hence it follows that the matrix

$$\hat{\mathcal{K}} = \begin{pmatrix} \hat{\rho} & \hat{\kappa} \\ \hat{\kappa}^* & I - \hat{\rho}^* \end{pmatrix} \tag{50}$$

satisfies the requirement

$$\hat{\mathcal{K}} \times \hat{\mathcal{K}} = \hat{\mathcal{K}}, \tag{51}$$

and, therefore, is the density matrix.

Thus, if the boson operators are replaced by c-numbers, then the images of the bifermion operators can be interpreted as elements of the density matrix $\hat{\mathcal{K}}$ of the Hartree–Fock–Bogoliubov theory. In particular, $\hat{\rho}$ can be identified with the one-particle density matrix and $\hat{\kappa}$ with the pairing tensor. When rotational motion is neglected, the Hartree–Fock–Bogoliubov energy functional is minimized with respect to the matrix elements of $\hat{\rho}$ and $\hat{\kappa}$, subject to the constraint $\hat{\mathcal{K}} \times \hat{\mathcal{K}} = \hat{\mathcal{K}}$. Since our boson parametrization of $\hat{\mathcal{K}}$ automatically satisfies this constraint, the solutions of the resulting equations

$$\frac{\partial H_{cc}}{\partial \Gamma_{\lambda k}(st)\,(\Delta_{\lambda k}(st))} = 0, \tag{52}$$

without further restrictions, correspond precisely to solutions of the Hartree-Fock-Bogoliubov variational equations [21].

It was indicated in [21] that replacement of the boson operators by c-numbers can be made in such a way that the quantal commutators of two operators A and B become Poisson brackets:

$$[A, B] \to \frac{1}{2} \sum_{sm,s'm'} \left(\frac{\partial A}{\partial \Delta_{sm,s'm'}} \frac{\partial B}{\partial \Gamma_{sm,s'm'}} - \frac{\partial B}{\partial \Delta_{sm,s'm'}} \frac{\partial A}{\partial \Gamma_{sm,s'm'}} \right). \tag{53}$$

Since the variational parameters $\Gamma_{sm,s'm'}$ and $\Delta_{sm,s'm'}$ are defined by Eq. (52), the commutator $[H_{cc}, \mathcal{K}]$ found in correspondence with Eq. (53) is equal to zero and, therefore, we obtain equations of the Hartree–Fock–Bogoliubov approximation.

However, in order to come to the self-consistent cranking model, the additional conditions imposed on $\Gamma_{\lambda k}(st)$ and $\Delta_{\lambda k}(st)$ should be taken into account. In the harmonic approximation, we can limit ourselves to the terms linear in bosonic operators in the expressions for the single particle operators. Then

$$\hat{n} = 2\sum_{\lambda kst} \Gamma_{\lambda k}(st)\Delta_{\lambda k}(st) + 2\sum_{\lambda kst}(\Delta_{\lambda k}(st)\beta^+_{\lambda k}(st) + \Gamma_{\lambda k}(st)\beta_{\lambda k}(st)) \tag{54}$$

and

$$\begin{aligned}\hat{L}_\eta = &\sum_{\lambda kk'st} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta} \Gamma_{\lambda k}(st)\Delta_{\lambda k'}(st) + \\ &+ \sum_{\lambda kk'st} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta}(\Delta_{\lambda k'}(st)\beta^+_{\lambda k}(st) + \Gamma_{\lambda k}(st)\beta_{\lambda k'}(st)).\end{aligned} \tag{55}$$

The first additional condition is given by the relation

$$N = 2\sum_{\lambda kst} \Gamma_{\lambda k}(st)\Delta_{\lambda k}(st). \tag{56}$$

This relation should be taken into account in the variational procedure of minimization of the Hamiltonian over $\Gamma_{\lambda k}(st)$ and $\Delta_{\lambda k'}(st)$ by the usual way using the Lagrange factor. The situation with the additional conditions following from the relations (45) is more complicated, because the operators $\hat{L}_\eta$ do not commute with each other. The problem is solved by constructing the Routhian

$$\tilde{H}_{cc} = H_{cc} - \Lambda \cdot 2\sum_{\lambda kst} \Gamma_{\lambda k}(st)\Delta_{\lambda k}(st) - \sum_\eta \Omega_\eta \sum_{\lambda kk'st} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta} \Gamma_{\lambda k}(st)\Delta_{\lambda k'}(st), \tag{57}$$

where Λ is the chemical potential, which is determined taking into account (56). In the general case, which includes the possibility of nuclear rotation about an axis tilted relative to the principal axes of the deformed nuclear mean field [20], $\vec{\Omega}$ is given by

$$\vec{\Omega} = (|\Omega| \sin\vartheta \sin\varphi, |\Omega| \sin\vartheta \cos\varphi, |\Omega| \cos\vartheta).$$

The minimization of the Routhian includes minimization with respect to ϑ and φ at fixed $|\Omega|$. The value of $|\Omega|$ is detemined by the value of the total angular momentum I by the relation [22]

$$I = \sqrt{\sum_\eta \bar{L}^2_\eta}, \tag{58}$$

where

$$\bar{L}_\eta = \sum_{\lambda k k' st} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta} \Gamma_{\lambda k}(st) \Delta_{\lambda k'}(st) \tag{59}$$

is an average of $\hat{L}_\eta$ over the boson vacuum.

In the case of the axial symmetry of the self-consistent nuclear mean field, one of the components of the angular momentum operator corresponding to the axis orthogonal to the symmetry axis is taken to be equal to the total angular momentum I. Two other components are taken to be equal to zero.

The conditions $[\hat{H}, \hat{n}] = 0$ and $[\hat{H}, \hat{L}_\eta] = 0$ tell us that the operators $\sum_{\lambda kst}(\Delta_{\lambda k}(st)\beta^+_{\lambda k}(st) + \Gamma_{\lambda k}(st)\beta_{\lambda k}(st))$ and $\sum_{\lambda kk'st} \sqrt{\lambda(\lambda+1)} C^{\lambda k}_{\lambda k' 1\eta}(\Delta_{\lambda k'}(st)\beta^+_{\lambda k}(st) + \Gamma_{\lambda k}(st)\beta_{\lambda k'}(st))$ generate zero energy modes, i.e., correspond to the unphysical modes and therefore should be excluded from consideration.

Thus, as a result of the canonical transformation, the angle ϕ and the Euler angles θ_l are separated from the total number of dynamic variables. Therefore, subsequent approximations will no longer violate gradient and rotational invariance. Additional conditions encountered in solving the problem exclude from consideration the dynamic variables describing unphysical modes.

3. Conclusion

The problem of extracting dynamic variables directly related to the symmetry of the Hamiltonian with respect to rotations in ordinary three-dimensional and phase spaces from a system of fermions is examined. It is demonstrated that the use of a boson representation of bifermion operators enables the solution of this problem through the canonical transformation of dynamic variables, as proposed by N. N. Bogoliubov. The finite boson representation allows for the derivation of a closed-form expression for the Hamiltonian. Furthermore, the non-hermiticity of the boson representation in the conventional metric does not pose any difficulties when solving the problem within the self-consistent mean-field approximation. The resulting Hamiltonian, combined with the additional conditions that must be considered, provides a justification for the application of the cranking model in describing the rotational excitations of nuclei.

Conflict of Interest

The authors declare no conflict of interest.

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