

Decays $\tau \rightarrow \pi\pi\eta\nu_\tau$ and $\tau \rightarrow \pi\eta\eta\nu_\tau$ in the extended Nambu–Jona-Lasinio model

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Abstract

In the framework of the extended Nambu–Jona-Lasinio model, the processes $\tau \rightarrow \pi\pi\eta(\eta')\nu_\tau$ and $\tau \rightarrow \pi\eta\eta(\eta')\nu_\tau$ are considered taking into account mesons in the ground and first radially excited intermediate states. It is shown that in the processes $\tau \rightarrow \pi\pi\eta(\eta')\nu_\tau$ the vector channel is dominant, and in the processes $\tau \rightarrow \pi\eta\eta(\eta')\nu_\tau$ the main contribution is given by the axial-vector channel. The scalar meson a_0 plays a dominant role in processes with two η mesons in the final state. The significance of the relative phase between the ground and first radially excited states for these processes is shown. The obtained results for the $\tau \rightarrow \pi\pi\eta\nu_\tau$ process are in satisfactory agreement with the recent experimental data from BaBar and CMD-3, which differ from the averaged values given in the PDG tables.

Keywords: chiral Lagrangians, τ decays

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1. Introduction

The processes related to the nonperturbative region of the theory of strong interactions are the objects of intensive research in modern electron–positron collider centers such as SLAC (BaBar), KEK (Belle), BEPC II (BES III), VEPP (SND, CMD-3) and others. It is important to note the current experiments of BES III, Belle II, and the planned super Charm-Tau factories that add relevance to the study of the processes of strong interactions. The planned τ -decay experiments at the lepton colliders, particularly at the super Charm-Tau factory, provide high statistics and accuracy. The study of processes of meson production in τ decays is a good laboratory for the analysis of strong interaction effects at low energies.

However, the theoretical description of the processes in this energy region causes difficulties related to the inapplicability of the QCD perturbation theory. The spectral functions and the meson τ -decay widths can be calculated on the basis of phenomenological models taking into account the quark structures of mesons and the chiral symmetry of strong interactions. Among such models one can note the chiral perturbation theory [1, 2] and the Nambu–Jona-Lasinio (NJL) model [3–13], and various extended versions of these models.

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The NJL model is based on the chiral symmetry of strong interactions that is partially broken by the current masses of the u , d and s quarks. When considering the η mesons, it is important to take into account the mixing of the light u and d quarks with a heavier s quark. Such mixing appears as a result of accounting for the gluon anomaly that is well described in the NJL model by using the 't Hooft interaction [14, 15]. In the description of the decays of the τ lepton with the production of the η/η' mesons, the effect of chiral symmetry breaking plays an important role.

One can note the recent success of the NJL model in the description of a series of four-particle τ decays with three pseudoscalar mesons with kaons and η mesons in the final states [16–18].

Currently, the process $\tau \rightarrow \pi\pi\eta\nu_\tau$ is actively studied experimentally. In recent experiments on this decay, the results have been obtained that differ from the values given in the PDG: $\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{BaBar}} = (1.63 \pm 0.08) \cdot 10^{-3}$ [19], $\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{CMD-3}} = (1.68 \pm 0.17) \cdot 10^{-3}$ [20], and $\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{PDG}} = (1.39 \pm 0.07) \cdot 10^{-3}$ [21]. This gives relevance to the theoretical study of processes of this type. The $\tau \rightarrow \pi\pi\eta(\eta')\nu_\tau$ decays are closely related to the electron–positron annihilation processes, since the vector channel is decisive in both cases. Therefore, the widths of the τ decays can be estimated from the e^+e^- data using the conservation of vector current hypothesis (CVC) [22]. However, for a deeper understanding of the internal structure of the decays, it is definitely of interest to provide direct calculations.

The present work is devoted to the description of the decays $\tau \rightarrow \pi\pi\eta(\eta')\nu_\tau$ and $\tau \rightarrow \pi\eta\eta(\eta')\nu_\tau$ in the NJL model. For this purpose, both the ground and first radially excited meson states are taken into account as intermediate resonances. Besides, when considering the processes $\tau \rightarrow \pi\eta\eta(\eta')\nu_\tau$, the scalar states are taken into account, the contribution of which turns out to be significant, unlike most other processes of this type. An important role is played by taking into account the phase factor, which determines the phase shift between the ground and first radially excited states of mesons. The presence of such a phase factor was established in experimental studies of meson production processes in e^+e^- annihilation [19, 20, 23].

2. Lagrangian of the NJL model

The extended NJL model used in the present work was formulated in [24, 25]. In this model, quark–meson Lagrangian was obtained as a result of the mixing of the ground and first radially excited meson states:

$$\Delta L_{\text{int}} = \bar{q} \left[\sum_{j=\pm} \lambda_j^\rho (A_{a_0} a_0^j + B_{a_0} \hat{a}_0^j) + i\gamma^5 \sum_{j=\pm} \lambda_j^\pi (A_\pi \pi^j + B_\pi \hat{\pi}^j) + \frac{1}{2} \gamma^\mu \sum_{j=\pm} \lambda_j^\rho (A_\rho \rho_\mu^j + B_\rho \hat{\rho}_\mu^j) + \frac{1}{2} \gamma^\mu \gamma^5 \sum_{j=\pm} \lambda_j^\rho (A_{a_1} a_{1\mu}^j + B_{a_1} \hat{a}_{1\mu}^j) + i\gamma^5 \sum_{M=\eta, \eta'} \lambda_u a_M^u M \right] q, \quad (1)$$

where q is a triplet of the u , d and s quarks with the masses $m_u \approx m_d = m = 270$ MeV, $m_s = 420$ MeV; λ are the linear combinations of the Gell-Mann matrices. The first radially excited states are marked with a hat. The factors A_M and B_M are defined through the mixing angles:

$$\begin{aligned} A_M &= \frac{1}{\sin(2\theta_M^0)} [g_M \sin(\theta_M + \theta_M^0) + g'_M f(k_\perp^2) \sin(\theta_M - \theta_M^0)], \\ B_M &= \frac{-1}{\sin(2\theta_M^0)} [g_M \cos(\theta_M + \theta_M^0) + g'_M f(k_\perp^2) \cos(\theta_M - \theta_M^0)], \end{aligned} \quad (2)$$

Table 1. The values of the mixing angles of the ground and first radially excited mesons.

	π	ρ	a_1	a_0
θ_M	59.48°	81.80°	81.80°	72.0°
θ_M^0	59.12°	61.50°	61.50°	61.50°

where M designates the appropriate meson. The values of the mixing angles are given in Table 1.

In the case of the η mesons, the four states are mixed: η , η' , and their first radially excited states. In this regard, the factors a_η take the following form:

$$\begin{aligned} a_\eta^u &= 0.71g_\eta + 0.11g'_\eta f(k_\perp^2), \\ a_{\eta'}^u &= -0.32g_\eta - 0.48g'_\eta f(k_\perp^2). \end{aligned} \quad (3)$$

The scalar mesons are considered here as quark–antiquark chirally symmetric partners of the pseudoscalar mesons.

The form factor $f(k_\perp^2) = 1 + dk_\perp^2$ was included in the model to describe the first radially excited meson states; d is the slope parameter determined from the requirement that the inclusion of excited states does not change the quark condensate.

The coupling constants of the initial non-physical mesons with quarks appear as a result of the renormalization of the Lagrangian:

$$\begin{aligned} g_\pi &= g_\eta = \left(\frac{4}{Z_\pi} I_2 \right)^{-1/2}, & g'_\pi &= g'_\eta = \left(4I_2^{f^2} \right)^{-1/2}, \\ g_\rho &= g_{a_1} = \left(\frac{2}{3} I_2 \right)^{-1/2}, & g'_\rho &= g'_{a_1} = \left(\frac{2}{3} I_2^{f^2} \right)^{-1/2}, \\ g_{a_0} &= (4I_2)^{-1/2}, & g'_{a_0} &= \left(4I_2^{f^2} \right)^{-1/2}, \end{aligned} \quad (4)$$

where $Z_\pi = \left(1 - 6 \frac{m^2}{M_{a_1}^2} \right)^{-1}$ is the additional renormalization constant appearing while taking into account the π – a_1 transitions.

The integrals $I_n^{f^m}$ appear in quark loops during the renormalization of the Lagrangian and take the following form:

$$I_n^{f^m} = -i \frac{N_c}{(2\pi)^4} \int \frac{f^m(k_\perp^2)}{(m^2 - k_\perp^2)^n} \Theta(\Lambda_3^2 - k_\perp^2) d^4k, \quad (5)$$

where $\Lambda_3 = 1030$ MeV is the three-dimensional cutoff parameter [13].

3. The amplitude of the process $\tau \rightarrow \pi\pi\eta\nu_\tau$

The diagrams of the process $\tau \rightarrow \pi\pi\eta\nu_\tau$ are given in Figures 1 and 2.

The amplitude of this process in the NJL model takes the following form:

$$\begin{aligned} \mathcal{M} &= 8G_F V_{ud} m \{ \mathcal{M}_{c\rho} + \mathcal{M}_{\rho\rho} + \mathcal{M}_{c\hat{\rho}} + \mathcal{M}_{\rho\hat{\rho}} + \mathcal{M}_{\hat{\rho}\rho} + \mathcal{M}_{\hat{\rho}\hat{\rho}} + \\ &\quad + \mathcal{M}_{cb} + \mathcal{M}_{\rho b} + \mathcal{M}_{\hat{\rho}b} \} L^\mu e_{\mu\nu\lambda\delta} p_\eta^\nu p_{\pi^-}^\lambda p_{\pi^0}^\delta, \end{aligned} \quad (6)$$

where G_F is the Fermi constant, V_{ud} is the element of the Cabibbo–Kobayashi–Maskawa matrix, L^μ is the weak lepton current, and p_η , p_{π^-} and p_{π^0} are the momenta of the final mesons.

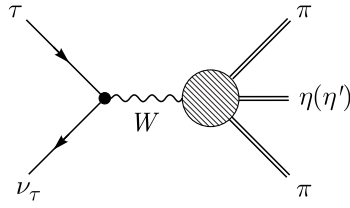


Figure 1. The contact diagram of the decays $\tau \rightarrow 2\pi\eta(\eta')\nu_\tau$. The shaded circle represents the sum of subdiagrams shown in Figure 3.

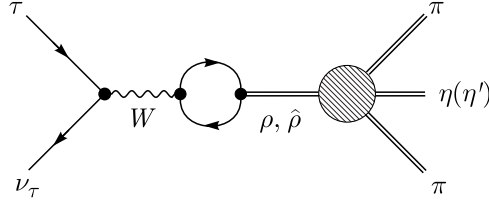


Figure 2. The diagram with the intermediate vector mesons ρ and $\hat{\rho} = \rho(1450)$ for τ decays. The shaded circle represents the sum of subdiagrams shown in Figure 3.

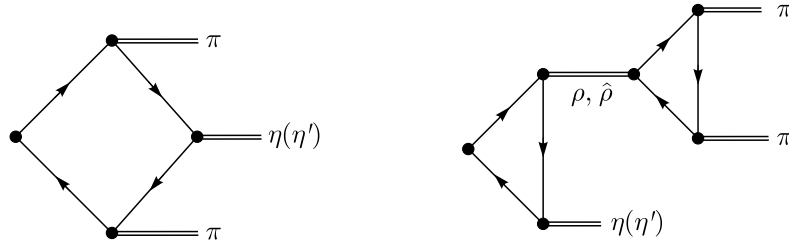


Figure 3. The vertices $V\pi\pi\eta(\eta')$ with the box quark diagram and two triangle quark diagrams connected by the virtual vector mesons ρ and $\hat{\rho}$.

The terms $\mathcal{M}_{\rho\rho}$, $\mathcal{M}_{\rho\hat{\rho}}$, $\mathcal{M}_{\hat{\rho}\rho}$ and $\mathcal{M}_{\hat{\rho}\hat{\rho}}$ describe the contributions from the diagrams with the intermediate ρ mesons in the ground and first radially excited states. The first and second indices refer to the first and second intermediate states. The index b denotes a box diagram. The index c denotes the contact contribution. These terms take the following form:

$$\begin{aligned}
 \mathcal{M}_{c\rho} &= 4I_3^{\rho\eta} I_2^{\rho\pi\pi} Z_{\rho\pi\pi} BW_{\rho}^{p_{\pi^-} + p_{\pi^0}}, \\
 \mathcal{M}_{\rho\rho} &= 4 \frac{C_{\rho}}{g_{\rho}} I_3^{\rho\rho\eta} I_2^{\rho\pi\pi} Z_{\rho\pi\pi} q^2 BW_{\rho}^q BW_{\rho}^{p_{\pi^-} + p_{\pi^0}}, \\
 \mathcal{M}_{c\hat{\rho}} &= 4I_3^{\hat{\rho}\eta} I_2^{\hat{\rho}\pi\pi} Z_{\hat{\rho}\pi\pi} BW_{\hat{\rho}}^{p_{\pi^-} + p_{\pi^0}}, \\
 \mathcal{M}_{\rho\hat{\rho}} &= 4 \frac{C_{\rho}}{g_{\rho}} I_3^{\rho\hat{\rho}\eta} I_2^{\hat{\rho}\pi\pi} Z_{\hat{\rho}\pi\pi} q^2 BW_{\rho}^q BW_{\hat{\rho}}^{p_{\pi^-} + p_{\pi^0}}, \\
 \mathcal{M}_{\hat{\rho}\rho} &= 4 \frac{C_{\hat{\rho}}}{g_{\rho}} I_3^{\rho\hat{\rho}\eta} I_2^{\rho\pi\pi} Z_{\rho\pi\pi} q^2 BW_{\hat{\rho}}^q BW_{\rho}^{p_{\pi^-} + p_{\pi^0}}, \\
 \mathcal{M}_{\hat{\rho}\hat{\rho}} &= 4 \frac{C_{\hat{\rho}}}{g_{\rho}} I_3^{\hat{\rho}\hat{\rho}\eta} I_2^{\hat{\rho}\pi\pi} Z_{\hat{\rho}\pi\pi} q^2 BW_{\hat{\rho}}^q BW_{\hat{\rho}}^{p_{\pi^-} + p_{\pi^0}}, \\
 \mathcal{M}_{cb} &= I_4^{\pi\pi\eta}, \\
 \mathcal{M}_{\rho b} &= \frac{C_{\rho}}{g_{\rho}} I_4^{\rho\pi\pi\eta} q^2 BW_{\rho}^q, \\
 \mathcal{M}_{\hat{\rho} b} &= \frac{C_{\hat{\rho}}}{g_{\rho}} I_4^{\hat{\rho}\pi\pi\eta} q^2 BW_{\hat{\rho}}^q,
 \end{aligned} \tag{7}$$

where

$$Z_{\rho\pi\pi} = 1 - 4 \frac{I_2^{\rho a_1 \pi} I_2^{a_1 \pi}}{I_2^{\rho \pi \pi}} \frac{m^2}{M_{a_1}^2}. \quad (8)$$

Here the intermediate states are described by the Breit–Wigner propagator

$$BW_{\text{meson}}^p = \frac{1}{M_{\text{meson}}^2 - p^2 - i\sqrt{p^2}\Gamma_{\text{meson}}}. \quad (9)$$

The integrals that arise in the amplitude with meson vertices take the form

$$I_n^{M\dots\hat{M}\dots} = -i \frac{N_c}{(2\pi)^4} \int \frac{A_M \dots B_M \dots}{(m^2 - k^2)^n} \Theta(\Lambda_3^2 - k_\perp^2) d^4k, \quad (10)$$

where A and B are the factors from the Lagrangian (1) relating the quark and meson fields and containing the mixing angles.

The branching fraction of the process calculated using the amplitude given above turns out to be lower than the experimental data presented in the PDG:

$$\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{NJL}} = (0.75 \pm 0.11) \cdot 10^{-3}, \quad (11)$$

$$\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{exp}} = (1.39 \pm 0.07) \cdot 10^{-3} \text{ [21]}. \quad (12)$$

The theoretical uncertainty of the NJL model can be estimated approximately at the level of 15% [26, 27].

According to the works [19, 28], the phase factors $e^{i\phi}$ with $\phi = 180^\circ$ describing the phase shift between the ground and first radially excited states are of great importance. However, this factor is not described by the NJL model and can be included on the basis of experimental data by redefining the Breit–Wigner propagator for excited meson states:

$$BW_{\rho'}^p \rightarrow e^{i\phi} BW_{\rho'}^p. \quad (13)$$

Then the absence of this factor is appropriate for the case $\phi = 0^\circ$.

While taking into account $e^{i\pi}$, the branching fraction of this process in the NJL model turns out to be higher than the average value given in the PDG but close to recent BaBar and CMD-3 data:

$$\begin{aligned} \text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{NJL}} &= (1.84 \pm 0.27) \cdot 10^{-3} \quad (\phi = 180^\circ), \\ \text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{BaBar}} &= (1.63 \pm 0.08) \cdot 10^{-3} \text{ [19]}, \\ \text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau)_{\text{CMD-3}} &= (1.68 \pm 0.17) \cdot 10^{-3} \text{ [20]}. \end{aligned} \quad (14)$$

The theoretical result obtained with $e^{i\pi}$ is in agreement with the experimental data [19, 20] in the framework of the theoretical and experimental uncertainties. The phase in the range of $\phi = 150^\circ$ – 180° leads to values of the branching fractions $\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau) = 1.30 \cdot 10^{-3}$ – $1.84 \cdot 10^{-3}$. As was noted above, the NJL model does not describe the relative phase between different states. Therefore, in this paper we use the phase value in accordance with the BaBar and CMD-3 data: $\phi = 0^\circ$ and $\phi = 180^\circ$ [19, 28]. Note that the inclusion of the phase factor previously led to a noticeable improvement in the results obtained for the processes of τ decays and e^+e^- annihilation containing intermediate channels with radially excited mesons [26].

The structure of the amplitude of the decay $\tau \rightarrow \pi\pi\eta'\nu_\tau$ almost coincides with the structure of the amplitude of the decay $\tau \rightarrow \pi\pi\eta\nu_\tau$ with the replacement of vertices related to the η meson by the vertices related to the η' meson under the integrals.

The experimental limit on the branching fraction of this process is [21]

$$\text{Br}(\tau \rightarrow \pi\pi\eta'\nu_\tau)_{\text{PDG}} < 1.2 \cdot 10^{-5} \quad (15)$$

at the 90% confidence level.

The theoretical results in the case $\phi = 0^\circ$ satisfy this constraint:

$$\text{Br}(\tau \rightarrow \pi\pi\eta'\nu_\tau) = (0.32 \pm 0.04) \cdot 10^{-5} \quad (\phi = 0^\circ). \quad (16)$$

However, the use of the phase $\phi = 180^\circ$ leads to exceeding the experimental threshold:

$$\text{Br}(\tau \rightarrow \pi\pi\eta'\nu_\tau) = (2.45 \pm 0.36) \cdot 10^{-5} \quad (\phi = 180^\circ). \quad (17)$$

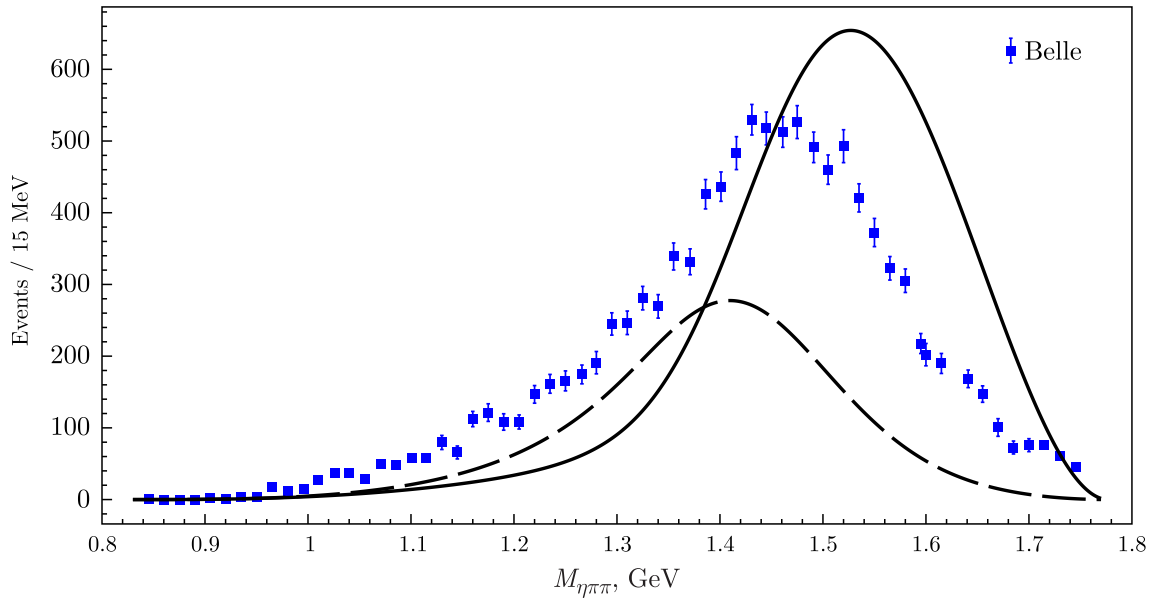


Figure 4. The invariant mass distribution for $\tau^- \pi\pi\eta\nu_\tau$ decay. The experimental data are taken from [29]. The solid line describes NJL result with the phase $\phi = 180^\circ$ corresponding to formula (13). The results with zero phase are shown by the dotted line.

The invariant mass distribution for the $\tau \rightarrow \pi\pi\eta\nu_\tau$ process and a comparison with the Belle data are shown in Figure 4. Our peak corresponding to the intermediate resonance $\rho'(1450)$ is shifted somewhat to the right compared to the Belle data [29]. In addition, this resonance is larger in magnitude than the experimental data. It is interesting to note that a similar situation for the differential decay width was obtained in [30] analyzing the $e^+e^- \rightarrow \pi\pi\eta$ and $\tau \rightarrow \pi\pi\eta\nu_\tau$ data using CVC.

4. The decay $\tau \rightarrow \pi^- 2\eta\nu_\tau$

The total decay amplitude of $\tau \rightarrow \pi^- 2\eta\nu_\tau$ consists of the contributions from the intermediate axial-vector and pseudoscalar channels in the ground and first radially excited states. For these channels, we take into account $a_0(980)$ as the second intermediate state. The total decay amplitude can be written as

$$\mathcal{M}(\tau \rightarrow \pi^- 2\eta\nu_\tau) = 64G_F V_{ud} L_\mu \{ \mathcal{M}_{ca_0} + \mathcal{M}_{a_1 a_0} + \mathcal{M}_{\hat{a}_1 a_0} + \mathcal{M}_{\pi a_0} + \mathcal{M}_{\hat{\pi} a_0} \}^\mu, \quad (18)$$

where the hadronic part of the amplitude is presented in curly brackets. Here the first three terms correspond to the contributions from the axial-vector channels, i.e., the contact channel and the channel with the axial-vector mesons a_1 and \hat{a}_1 in the first intermediate state. The last two terms correspond to the pseudoscalar channels. In all channels, the scalar meson $a_0(980)$ is represented as the second intermediate state, which decays into final products through the strong vertex $a_0\pi\eta$. The expressions for the hadronic parts of this amplitude of the contact, axial-vector, and pseudoscalar channels have the form

$$\begin{aligned}
\mathcal{M}_{ca_0} &= m_u I_2^{a_0\pi\eta} I_2^{a_0\eta} Z_{a_1\pi\eta} BW_{a_0}^{p_{\pi^-} + p_{\eta^{(1)}}} g_{\mu\nu} (p_{\pi^-} + p_{\eta^{(1)}} - p_{\eta^{(2)}})_\nu + (p_{\eta^{(1)}} \leftrightarrow p_{\eta^{(2)}}), \\
\mathcal{M}_{a_1a_0} &= m_u \frac{C_{a_1}}{g_\rho} I_2^{a_0\pi\eta} I_2^{a_1a_0\eta} Z_{a_1\pi\eta} BW_{a_0}^{p_{\pi^-} + p_{\eta^{(1)}}} (g_{\mu\nu} (q^2 - 6m_u^2) - q_\mu q_\nu) \times \\
&\quad \times BW_{a_1}^q (p_{\pi^-} + p_{\eta^{(1)}} - p_{\eta^{(2)}})_\nu + (p_{\eta^{(1)}} \leftrightarrow p_{\eta^{(2)}}), \\
\mathcal{M}_{\hat{a}_1a_0} &= m_u \frac{C_{\hat{a}_1}}{g_\rho} I_2^{a_0\pi\eta} I_2^{\hat{a}_1a_0\eta} Z_{a_1\pi\eta} BW_{a_0}^{p_{\pi^-} + p_{\eta^{(1)}}} (g_{\mu\nu} (q^2 - 6m_u^2) - q_\mu q_\nu) \times \\
&\quad \times BW_{\hat{a}_1}^q (p_{\pi^-} + p_{\eta^{(1)}} - p_{\eta^{(2)}})_\nu + (p_{\eta^{(1)}} \leftrightarrow p_{\eta^{(2)}}), \\
\mathcal{M}_{\pi a_0} &= 4F_\pi I_2^{a_0\pi\eta} I_2^{a_0\pi\eta} Z_{\pi\pi\eta} BW_\pi^q BW_{a_0}^{p_{\pi^-} + p_{\eta^{(1)}}} q^\mu + (p_{\eta^{(1)}} \leftrightarrow p_{\eta^{(2)}}), \\
\mathcal{M}_{\hat{\pi} a_0} &= 4F_\pi I_2^{\hat{\pi}a_0\eta} I_2^{a_0\pi\eta} Z_{\hat{\pi}\pi\eta} BW_{\hat{\pi}}^q BW_{a_0}^{p_{\pi^-} + p_{\eta^{(1)}}} q^\mu + (p_{\eta^{(1)}} \leftrightarrow p_{\eta^{(2)}}),
\end{aligned} \tag{19}$$

where $q = p_\pi + p_{\eta^{(1)}} + p_{\eta^{(2)}}$. Here the factors $Z_{a_1\pi\eta}$ and $Z_{\pi\pi\eta}$ arise when taking into account the non-diagonal π - a_1 and η - f_1 transitions. The expressions for these factors take the form

$$Z_{a_1\pi\eta} = 1 - \frac{I_2^{a_1a_0\eta} I_2^{a_1\pi} (p_{\pi^-} + p_{\eta^{(1)}})^2 - M_\eta^2}{I_2^{a_0\pi\eta} M_{a_1}^2} - \frac{I_2^{f_1a_0\pi} I_2^{f_1\eta} (p_{\pi^-} + p_{\eta^{(1)}})^2 - M_\eta^2}{I_2^{a_0\pi\eta} M_{f_1}^2}, \tag{20}$$

$$\begin{aligned}
Z_{\pi\pi\eta} &= \left(1 - \frac{I_2^{f_1a_0\pi} I_2^{f_1\eta} q^2 - (p_{\pi^-} + p_{\eta^{(1)}})^2}{I_2^{a_0\pi\eta} M_{f_1}^2} - \frac{I_2^{a_1a_0\eta} I_2^{a_1\pi} (p_{\pi^-} + p_{\eta^{(1)}})^2 - M_\eta^2}{I_2^{a_0\pi\eta} M_{a_1}^2} \right) \times \\
&\quad \times \left(1 - \frac{I_2^{a_1a_0\eta} I_2^{f_1\eta} (p_{\pi^-} + p_{\eta^{(1)}})^2 - M_\eta^2}{I_2^{a_0\pi\eta} M_{f_1}^2} - \frac{I_2^{f_1a_0\pi} I_2^{f_1\eta} (p_{\pi^-} + p_{\eta^{(1)}})^2 - M_\pi^2}{I_2^{a_0\pi\eta} M_{f_1}^2} \right).
\end{aligned} \tag{21}$$

As a result, in the extended NJL model for the branching fractions of decay $\tau \rightarrow \pi^- 2\eta \nu_\tau$ we obtain

$$\begin{aligned}
\text{Br}(\tau \rightarrow \pi\eta\eta\nu_\tau) &= (6.0 \pm 0.9) \cdot 10^{-6} \quad (\phi = 180^\circ), \\
\text{Br}(\tau \rightarrow \pi\eta\eta\nu_\tau) &= (8.80 \pm 1.32) \cdot 10^{-6} \quad (\phi = 0^\circ).
\end{aligned} \tag{22}$$

The experiment only provides an upper limit on the decay branching fractions [21]

$$\text{Br}(\tau \rightarrow \pi\eta\eta\nu_\tau)_{\text{PDG}} = < 7.4 \cdot 10^{-6}. \tag{23}$$

The result in the NJL model for the phase $\phi = 180^\circ$ satisfies these data.

The decay amplitude of $\tau \rightarrow \pi\eta\eta'\nu_\tau$ can be obtained from formula (18) with the replacement of the vertices $\eta \rightarrow \eta'$ and the mass $M_\eta \rightarrow M_{\eta'}$, respectively. The calculations within the extended NJL model lead to the following result for the decay branching fractions $\tau \rightarrow \pi\eta\eta'\nu_\tau$

$$\text{Br}(\tau \rightarrow \pi\eta\eta'\nu_\tau) = (4.0 \pm 0.6) \cdot 10^{-7}. \tag{24}$$

5. Conclusion

In this paper, three-meson τ decays with the participation of η/η' mesons in the final state have been considered within the extended NJL model. The amplitudes for the contact channels and channels with intermediate vector, axial-vector and pseudoscalar mesons in the ground and first radially excited states were calculated. The results are obtained with the phase angle between the ground and first radially excited meson states. In the case of the $\tau \rightarrow \pi\pi\eta\nu_\tau$ process, the angle value $\phi = 180^\circ$ leads to a result that does not agree with the PDG data but is consistent with the recent results of the BaBar and CMD-3 collaborations [19, 20]. Calculation of the process $\tau \rightarrow \pi\pi\eta'\nu_\tau$ with a given value of the angle leads to exceeding the threshold given in PDG. However, there is reason to believe that further experimental study of this process may lead to a correction of this threshold, since analysis of the latest BaBar and CMD-3 data provides branching fractions somewhat larger than the average value given in PDG.

The decays $\tau \rightarrow \pi\pi\eta\nu_\tau$ and $\tau \rightarrow \pi\pi\eta'\nu_\tau$ occur only through vector channels. These decays were previously described in the NJL model in the paper [31]. However, the decay width can be calculated based on the e^+e^- -annihilation process using the phase volume transformation. Similar calculations of the decays $\tau \rightarrow \pi\pi\eta(\eta')\nu_\tau$ within the CVC hypothesis were performed in the paper [22] where the decay widths were estimated as $\text{Br}(\tau \rightarrow \pi\pi\eta\nu_\tau) = (1.53 \pm 0.18) \cdot 10^{-3}$ and $\text{Br}(\tau \rightarrow \pi\pi\eta'\nu_\tau) < 3.2 \cdot 10^{-5}$.

The $\tau \rightarrow \pi\pi\eta\nu_\tau$ process was considered in the theoretical work [32] within the chiral perturbation theory with resonances. However, the calculations were performed without taking into account the contribution of the intermediate radially excited $\rho(1450)$ meson. The spectral function for the invariant mass $M_{\pi\pi\eta}$ was fitted by model parameters based on the Belle data [29]. At the same time, the calculations in the present work in the NJL model show that the contribution from the intermediate $\rho(1450)$ meson is important for describing the decay widths.

In the paper [32] the $\tau \rightarrow \pi\pi\eta'\nu_\tau$ decay was also considered. However, the theoretical estimate for the partial width of this decay is in the range $\text{Br}(\tau \rightarrow \pi\pi\eta'\nu_\tau) = (1 - 4.5) \cdot 10^{-4}$ and exceeds the experimental threshold. It is important to note that the estimates for the $\tau \rightarrow \pi\pi\eta'\nu_\tau$ decay obtained using the CVC hypothesis and within the NJL model exceed the experimental limit for the decay width. All this may indicate the need for a more thorough experimental and theoretical study of this process.

The results of calculations in the NJL model for the decays $\tau \rightarrow \pi\eta\eta(\eta')\nu_\tau$ are given in Section 4. In contrast to the above processes, the matrix elements of these decays are determined by the axial current. The calculations show that the main contribution to the decay width of $\tau \rightarrow \pi\eta\eta\nu_\tau$ is given by the axial-vector channels with the intermediate states $a_1(1260)$ and $a_1(1640)$ containing the scalar meson $a_0(980)$. Note that among the numerous τ decays, the decay $\tau \rightarrow \pi\eta\eta\nu_\tau$ is unique where the scalar isovector meson a_0 plays a dominant role in determining the decay width.

In a recent paper [33], a theoretical estimation of the contribution of the intermediate scalar meson $f_0(500)$ to the total decay width $\tau \rightarrow \pi^-\pi^0\pi^0\nu_\tau$ was given. It was shown that the contribution of the intermediate channel $f_0(500)\pi$ is 13% of the total decay width, which is consistent with the CLOE II data $(16.18 \pm 3.85) \%$ [34]. Moreover, taking into account the scalar channel in this decay turns out to be important for a correct description of the invariant mass distribution function of three pions. All this demonstrates the importance of taking into account channels with intermediate scalar mesons for a number of three-meson τ decays.

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Conflicts of Interest

The authors declare no conflicts of interest.

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