NATURAL SCIENCE REVIEW **2** 100204 (2025)
nsr.jinr.int

# Quantum groups and Yang–Baxter equations

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**Abstract**

This introductory review is devoted to the newest section of the theory of symmetries — the theory of quantum groups. The principles of the theory of quantum groups are reviewed from the point of view of the possibility of their use for deformations of symmetries in physics models. The $R$-matrix approach to the theory of quantum groups is discussed in detail and is taken as the basis of the quantization of classical Lie groups, as well as some Lie supergroups. We start by laying out the foundations of noncommutative and noncocommutative Hopf algebras. Much attention has been paid to the Hecke and Birman–Murakami–Wenzl (BMW) $R$-matrices and related quantum matrix algebras. Noncommutative differential geometry on quantum groups of special types is discussed. Trigonometric solutions of the Yang–Baxter equations associated with the quantum groups $GL_q(N)$, $SO_q(N)$, $Sp_q(2n)$ and supergroups $GL_q(N|M)$, $Osp_q(N|2m)$, as well as their rational (Yangian) limits, are presented. Rational $R$-matrices for exceptional Lie algebras and elliptic solutions of the Yang–Baxter equation are also considered. The basic concepts of the group algebra of the braid group and its finite-dimensional quotients (such as the Hecke and BMW algebras) are outlined. A sketch of the representation theories of the Hecke and BMW algebras is given, including methods for finding idempotents (quantum Young projectors) and their quantum dimensions. Applications of the theory of quantum groups and Yang–Baxter equations in various areas of theoretical physics are briefly discussed.

This is a modified version of the review paper published in 2004 as a preprint [47] of the Max-Planck-Institut für Mathematik (MPIM) in Bonn.

*Keywords:* Noncommutative differential geometry, quantum groups, Yang–Baxter equations, Feynman diagrams, spin chains, braid groups, knot theory

## Contents



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Received: 10 February 2025; Revised: 6 March 2025; Accepted: 6 March 2025; Published: 31 March 2025

## 1. Introduction

In modern theoretical and mathematical physics, the ideas of symmetry and invariance play a very important role. Sets of symmetry transformations form groups, and therefore the most natural language for describing symmetries is the group theory language.

About 40 years ago, in the study of quantum integrable systems [1–5], in particular, in the framework of the quantum inverse scattering method [6–9], new algebraic structures arose, the generalizations of which were later called quantum groups [10][1]. Yang–Baxter equations (which firstly appeared and were used in [2, 3]) became a unifying basis of all these investigations. The history of the creation of the quantum inverse scattering method and the origin of the term "Yang–Baxter equation" are described in the review [11], Section 5.

The most important nontrivial examples of quantum groups are quantizations (or deformations) of ordinary classical Lie groups and algebras (more precisely, one considers the deformations of the algebra of functions of a Lie group and the universal enveloping of a Lie algebra; see, e.g., [113, 114]). The quantization is accompanied by the introduction of an additional parameter $q$ (the deformation parameter), which plays a role analogous to the role of Planck's

[1] In pure mathematics, analogous structures appeared as nontrivial examples of "ring-groups" introduced by G. Kac; see, e.g., [62] and references therein.

constant in quantum mechanics. In the special limit $q \to 1$, the quantum groups and algebras go over into the classical ones. Although quantum groups are deformations of the usual groups, they nevertheless still possess several properties that make it possible to speak of them as of "symmetry groups". Moreover, one can claim that the quantum groups serve as symmetries and provide integrability in exactly solvable quantum systems (see, e.g., [12–14])[2]. In this connection, the idea naturally arises of looking for and constructing other physical models possessing such quantum symmetries. Some of the realizations of this idea use the similarity of the representation theories of quantum and classical Lie groups and algebras (for $q$ not equal to the root of unity). Thus, there were attempts to apply quantum groups and algebras in the classification and phenomenology of elementary particles and in nuclear spectroscopy investigations. Further, it is natural to investigate already existing field-theoretical models (especially gauge quantum field theories) from the point of view of relations (see, e.g., [16, 17]) to the noncommutative geometry [15][3] and, in particular, the possibility of their invariance with respect to quantum-group transformations. An attractive idea is that of relating the deformation parameters of quantum groups to the mixing angles that occur in the Standard Model as free parameters. One of the possible realizations of this idea was proposed in [21] (see also [22]). Of course, it is necessary to mention here numerous attempts to deform Lorentz and Poincaré groups and construct a covariant quantum Minkowski space–time corresponding to these deformations [23–31]. Finally, promising studies of Yangian symmetries of planar Feynman graphs and scattering amplitudes in supersymmetric Yang–Mills theories should be noted (see, e.g., [32–40]; see also the review [41] and references therein).

It is clear that the approaches listed above (associated with deformations of symmetries in physics) present only a small fraction of all the applications of the theory of quantum groups. Quantum groups and Yang–Baxter equations naturally arise in many problems of theoretical physics, and this makes it possible to speak of them and their theories as of an important paradigm in mathematical physics. Unfortunately, the strict limits of this review make it impossible to discuss in detail all applications of quantum groups and Yang–Baxter equations. We have therefore restricted ourselves to a brief listing of certain areas in theoretical and mathematical physics in which quantum groups and Yang–Baxter equations play an important role. The incomplete list is given in Section **5**. In Section **2**, the mathematical foundations of the theory of quantum groups are outlined. A significant part of Section **3** is a detailed exposition of the results of the famous work by Faddeev, Reshetikhin, and Takhtajan [42] who formulated the $R$-matrix approach to the theory of quantum groups. In Section **3**, we also consider $R$-matrix formulation of link and knot invariants, problems of invariant Baxterization of $R$-matrices, multiparameter deformations of Lie groups, the quantization of some Lie supergroups, and various aspects of differential geometry on special types of quantum groups. The rational solutions of the Yang–Baxter equation for exceptional Lie algebras are also considered in Section **3**. At the end of Section **3**, we present the basic notions of the theory of quantum Knizhnik–Zamolodchikov equations and discuss elliptic solutions of the Yang–Baxter equation for which the algebraic basis (the type of quantum universal enveloping Lie algebras $U_q(g)$ in the case of trigonometric solutions) has not yet been completely clarified (see, however, [43–45]). In Section **4**, we briefly discuss the group algebra of a braid group and its finite-dimensional quotients such as the Hecke and Birman–Murakami–Wenzl (BMW) algebras. The Hecke and BMW algebras are, respectively, quantum analogs of the group algebra of the permutation

[2] The Yangian symmetries are the symmetries of the same type.

[3] After the quantization of any Poisson manifold [18] and the appearance of papers [19, 20], the subject of field theories on noncommutative spaces became very popular from the point of view of string theories.

group and the Brauer algebra. A sketch of the representation theories of the Hecke and BMW algebras is given (including methods for finding idempotents and their quantum dimensions). The representation theories of the Hecke and BMW algebras are important for understanding of the quantum analog of Schur–Weyl and Schur–Weyl–Brauer dualities for linear and orthogonal/simplectic quantum groups. The part of the content of Section **4** can be considered as a different presentation of some facts from Section **3**. In Section **5**, some applications of quantum groups and Yang–Baxter equations are outlined.

This introductory review is based on the paper [46] and the MPIM (Bonn) preprint [47] published in 1995 and 2004, respectively. An extended version of the preprint [47] is available in [48]. Comparing to the previous version [47], this text has been considerably changed only in Subsections **3.1**, **3.2**, **3.4.3**, **3.8**, **4.3.6**, and **5.3**. The structure of the review has also been significantly modified. Three new Subsections **3.5.3**, **3.13**, and **4.5** have been added, and Subsection **4.4** has been extended. In the course of the presentation, we have also tried briefly mention some of the new results. According to this, we have refreshed the list of references. Among the added references, we highlight the paper [11], where Section 6 provides a brief overview of the quantum group theory, including a discussion of the quantum dilogarithm and Faddeev's modular double.

## 2. Hopf algebras

### *2.1. Coalgebras*

We consider an associative unital algebra $\mathcal{A}$ (over the field of complex numbers $\mathbb{C}$; in what follows, all algebras that are introduced will also be understood to be over the field of complex numbers). Each element of $\mathcal{A}$ can be expressed as a linear combination of basis elements $e_i \in \mathcal{A}$, where $i = 1, 2, 3, \dots$, and the identity element $I$ is given by the formula

$$I = E^i \, e_i \qquad (E^i \in \mathbb{C})$$

(we imply summation over repeated indices). Then for any two elements $e_i$ and $e_j$, we define their multiplication in the form

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \quad \Rightarrow \quad e_i \cdot e_j = m_{ij}^k e_k, \tag{2.1.1}$$

where $m_{ij}^k$ is a certain set of complex numbers that satisfy the condition

$$E^i m_{ij}^k = m_{ji}^k E^i = \delta_j^k \tag{2.1.2}$$

for the identity element, and also the condition

$$m_{ij}^l m_{lk}^n = m_{il}^n m_{jk}^l \equiv m_{ijk}^n \tag{2.1.3}$$

that is equivalent to the condition of associativity for the algebra $\mathcal{A}$:

$$(e_i e_j) e_k = e_i (e_j e_k). \tag{2.1.4}$$

The condition of associativity (2.1.4) for the multiplication (2.1.1) can obviously be represented in the form of the commutativity of the diagram in Figure 1:

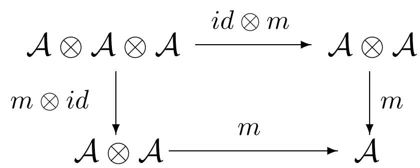



**Figure 1.** Associativity axiom.

In Figure 1, the map $m$ represents multiplication: $\mathcal{A}\otimes\mathcal{A}\xrightarrow{m}\mathcal{A}$, and $id$ denotes the identity mapping. The existence of the unit element $I$ means that one can define a mapping $\mathbf{i}$: $\mathbb{C}\to\mathcal{A}$ (embedding of $\mathbb{C}$ in $\mathcal{A}$)

$$k\xrightarrow{\mathbf{i}}k\cdot I,\quad k\in\mathbb{C}. \tag{2.1.5}$$

For $I$ we have the condition (2.1.2), which is visualized as the diagram in Figure 2:

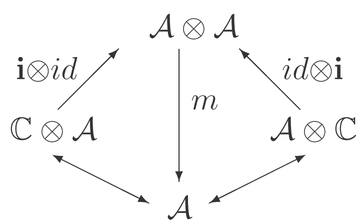



**Figure 2.** Axioms for the identity.

Here the mappings

$$\mathbb{C}\otimes\mathcal{A}\leftrightarrow\mathcal{A}\quad\text{and}\quad\mathcal{A}\otimes\mathbb{C}\leftrightarrow\mathcal{A} \tag{2.1.6}$$

are natural isomorphisms. One of the advantages of the diagrammatic language used here is that it leads directly to the definition of a new fundamental object — the coalgebra — if we reverse all the arrows in the diagrams of Figures 1 and 2.

**Definition 1.** *A coalgebra $\mathcal{C}$ is a vector space (with the basis $\{e_i\}$) equipped with the mapping $\Delta:\mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$*

$$\Delta(e_i)=\Delta_i^{kj}e_k\otimes e_j, \tag{2.1.7}$$

*which is called the comultiplication, and also equipped with the mapping $\epsilon:\mathcal{C}\to\mathbb{C}$, which is called the coidentity. The coalgebra $\mathcal{C}$ is called coassociative if the mapping $\Delta$ satisfies the condition of coassociativity (cf. the diagram in Figure 1 with the arrows reversed and the symbol $m$ changed to $\Delta$)*

$$(id\otimes\Delta)\Delta=(\Delta\otimes id)\Delta\quad\Rightarrow\quad\Delta_i^{nl}\Delta_l^{kj}=\Delta_i^{lj}\Delta_l^{nk}\equiv\Delta_i^{nkj}. \tag{2.1.8}$$

*The coidentity $\epsilon$ must satisfy the following conditions (cf. the diagram in Figure 2 with arrows reversed and symbols $m,\mathbf{i}$ changed to $\Delta,\epsilon$)*

$$m\big((\epsilon\otimes id)\Delta(\mathcal{C})\big)=m\big((id\otimes\epsilon)\Delta(\mathcal{C})\big)=\mathcal{C}\quad\Rightarrow\quad\epsilon_i\Delta_k^{ij}=\Delta_k^{ji}\epsilon_i=\delta_k^j. \tag{2.1.9}$$

*Here $m$ realizes the natural isomorphisms (2.1.6) as a multiplication map: $m(c\otimes e_i)=m(e_i\otimes c)=c\cdot e_i$ ($\forall c\in\mathbb{C}$), and the complex numbers $\epsilon_i$ are determined from the relations $\epsilon(e_i)=\epsilon_i$.*

For algebras and coalgebras, the concepts of modules and comodules can be introduced. Thus, if $\mathcal{A}$ is an algebra, the left $\mathcal{A}$-module can be defined as a vector space $N$ and a mapping $\psi$: $\mathcal{A}\otimes N\to N$ (action of $\mathcal{A}$ on $N$) such that the diagrams in Figure 3 are commutative.

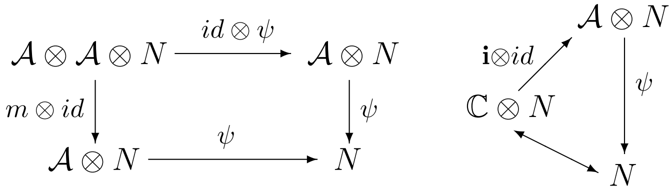



**Figure 3.** Axioms for the left $\mathcal{A}$-module.

In other words, the space $N$ is the representation space of the algebra $\mathcal{A}$.

If $N$ is a (co)algebra and the mapping $\psi$ preserves the (co)algebraic structure of $N$ (see below), then $N$ is called the *left $\mathcal{A}$-module (co)algebra*. The concept of the *right module (co)algebra* is introduced similarly. If $N$ is simultaneously the left and the right $\mathcal{A}$-module, then $N$ is called the *two-sided $\mathcal{A}$-module*. It is obvious that the algebra $\mathcal{A}$ itself is a two-sided $\mathcal{A}$-module for which the left and right actions are given by the left and right multiplications in the algebra.

Now suppose that $\mathcal{C}$ is a coalgebra; then a left $\mathcal{C}$-comodule can be defined as a space $M$ together with a mapping $\Delta_L$: $M \to \mathcal{C}\otimes M$ (coaction of $\mathcal{C}$ on $M$) satisfying the axioms in Figure 4 (in the diagrams of Figure 3, where the modules were defined, it is necessary to reverse all the arrows):

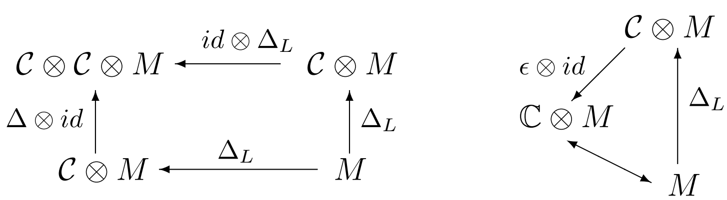



**Figure 4.** Axioms for the left $\mathcal{A}$-comodule.

If $M$ is a (co)algebra and the mapping $\Delta_L$ preserves the (co)algebraic structure (for example, is a homomorphism; see below), then $M$ is called a left $\mathcal{C}$-comodule (co)algebra. The right comodules are introduced similarly, after which two-sided comodules are defined in the natural manner. It is obvious that the coalgebra $\mathcal{C}$ is a two-sided $\mathcal{C}$-comodule.

Let $\mathcal{V},\ \tilde{\mathcal{V}}$ be two vector spaces with bases $\{e_i\},\ \{\tilde{e}_i\}$. We denote by $\mathcal{V}^*,\ \tilde{\mathcal{V}}^*$ the corresponding dual linear spaces whose basis elements are linear functionals $\{e^i\}:\ \mathcal{V}\to\mathbb{C}$, $\{\tilde{e}^i\}:\tilde{\mathcal{V}}\to\mathbb{C}$. For the values of these functionals, we use the expressions $\langle e^i\,|e_j\rangle$ and $\langle\tilde{e}^i\,|\tilde{e}_j\rangle$. For every mapping $L$: $\mathcal{V}\to\tilde{\mathcal{V}}$ it is possible to define a unique mapping $L^*:\ \ \tilde{\mathcal{V}}^*\to\mathcal{V}^*$ induced by the equations

$$\langle\tilde{e}^i\,|L(e_j)\rangle = \langle L^*(\tilde{e}^i)\,|e_j\rangle, \tag{2.1.10}$$

if the matrix $\langle e^i\,|e_j\rangle$ is invertible. In addition, for the dual objects there exists the linear injection

$$\rho:\ \ \mathcal{V}^*\otimes\tilde{\mathcal{V}}^*\to(\mathcal{V}\otimes\tilde{\mathcal{V}})^*,$$

which is given by the equations

$$\langle\rho(e^i\otimes\tilde{e}^j)\,|e_k\otimes\tilde{e}_l\rangle = \langle e^i\,|e_k\rangle\ \langle\tilde{e}^j\,|\tilde{e}_l\rangle.$$

A consequence of these facts is that for every coalgebra $(\mathcal{C},\ \Delta,\ \epsilon)$, it is possible to define an algebra $\mathcal{C}^*=\mathcal{A}$ (as dual object to $\mathcal{C}$) with multiplication $m=\Delta^*\cdot\rho$ and the unit element $I$ that satisfy the relations ($\forall a,a'\in\mathcal{A}$, $\forall c\in\mathcal{C}$):

$$\langle a|c_{(1)}\rangle\langle a'|c_{(2)}\rangle = \langle\rho(a\otimes a')|\Delta(c)\rangle = \langle\Delta^*\cdot\rho(a\otimes a')|c\rangle = \langle a\cdot a'\,|\,c\rangle, \qquad \langle I|c\rangle=\epsilon(c).$$

Here we denote $a \cdot a' := \Delta^* \cdot \rho(a \otimes a')$ and use the convenient Sweedler notation of [11] for comultiplication in $\mathcal{C}$ (cf. Eq. (2.1.7)):

$$\Delta(c) = \sum_c c_{(1)} \otimes c_{(2)} . \qquad (2.1.11)$$

The summation symbol $\sum_c$ will usually be omitted in the equations. We also use the Sweedler notation for the left and right coactions $\Delta_L(v) = \bar{v}^{(-1)} \otimes v^{(0)}$ and $\Delta_R(v) = v^{(0)} \otimes \bar{v}^{(1)}$, where index (0) is reserved for the comodule elements and summation symbols $\sum_v$ are also omitted.

Thus, duality in the diagrammatic definitions of the algebras and coalgebras (reversal of the arrows) has, in particular, the consequence that the algebras and coalgebras are indeed duals to each other.

It is natural to expect that an analogous duality can also be traced for modules and comodules. Let $\mathcal{V}$ be a left comodule for $\mathcal{C}$. Then the left coaction of $\mathcal{C}$ on $\mathcal{V}$: $v \mapsto \sum_v \bar{v}^{(-1)} \otimes v^{(0)}$ $(\bar{v}^{(-1)} \in \mathcal{C},\ v^{(0)} \in \mathcal{V})$ induces the right action of $\mathcal{C}^* = \mathcal{A}$ on $\mathcal{V}$:

$$(v, a) \quad \mapsto \quad v \triangleleft a = \langle a \,|\bar{v}^{(-1)}\rangle \, v^{(0)}, \qquad a \in \mathcal{A},$$

and therefore $\mathcal{V}$ is a right module for $\mathcal{A}$. Conversely, the right coaction of $\mathcal{C}$ on $\mathcal{V}$: $v \mapsto v^{(0)} \otimes \bar{v}^{(1)}$ induces the left action of $\mathcal{A} = \mathcal{C}^*$ on $\mathcal{V}$:

$$(a, v) \quad \mapsto \quad a \triangleright v = v^{(0)} \langle a | \bar{v}^{(1)} \rangle .$$

From this we immediately conclude that the coassociative coalgebra $\mathcal{C}$ (which coacts on itself by the coproduct) is a natural module for its dual algebra $\mathcal{A} = \mathcal{C}^*$. Indeed, the right action $\mathcal{C} \otimes \mathcal{A} \to \mathcal{C}$ is determined by the equations

$$(c, a) \quad \mapsto \quad c \triangleleft a = \langle a | c_{(1)} \rangle c_{(2)} , \qquad (2.1.12)$$

whereas for the left action $\mathcal{A} \otimes \mathcal{C} \to \mathcal{C}$ we have

$$(a, c) \quad \mapsto \quad a \triangleright c = c_{(1)} \langle a | c_{(2)} \rangle . \qquad (2.1.13)$$

Here $a \in \mathcal{A}$ and $c \in \mathcal{C}$. The module axioms (shown as the diagrams in Figure 3) hold by virtue of the coassociativity of $\mathcal{C}$.

Finally, we note that the action of a certain algebra $H$ on $\mathcal{C}$ from the left (from the right) induces an action of $H$ on $\mathcal{A} = \mathcal{C}^*$ from the right (from the left). This obviously follows from relations of the type (2.1.10).

### *2.2. Bialgebras*

So-called bialgebras are the next important objects that are used in the theory of quantum groups.

**Definition 2.** *An associative algebra $\mathcal{A}$ with identity that is simultaneously a coassociative coalgebra with coidentity is called a bialgebra if the algebraic and coalgebraic structures are self-consistent. Namely, the comultiplication and coidentity must be homomorphisms of the algebras:*

$$\Delta(e_i) \, \Delta(e_j) = m_{ij}^k \Delta(e_k) \Rightarrow \Delta_i^{i'i''} \Delta_j^{j'j''} m_{i'j'}^{k'} m_{i''j''}^{k''} = m_{ij}^k \, \Delta_k^{k'k''} , \qquad (2.2.1)$$

$$\Delta(I) = I \otimes I, \quad \epsilon(e_i \cdot e_j) = \epsilon(e_i) \, \epsilon(e_j), \quad \epsilon(I) = E^i \, \epsilon_i = 1.$$

Note that for every bialgebra we have a certain freedom in the definition of the multiplication (2.1.1) and the comultiplication (2.1.7). Indeed, all the axioms (2.1.3), (2.1.8), and (2.2.1) are satisfied if instead of (2.1.1) we take

$$e_i \cdot e_j = m^k_{ji}\, e_k,$$

or instead of (2.1.7) we choose

$$\Delta^{\mathsf{op}}(e_i) = \Delta^{jk}_i\, e_k \otimes e_j \tag{2.2.2}$$

(such algebras are denoted as $\mathcal{A}^{\text{op}}$ and $\mathcal{A}^{\text{cop}}$, respectively). Then the algebra $\mathcal{A}$ is called noncommutative if $m^k_{ij} \neq m^k_{ji}$, and noncocommutative if $\Delta^{ij}_k \neq \Delta^{ji}_k$.

In quantum physics, it is usually assumed that all algebras of observables are bialgebras. Indeed, a coalgebraic structure is needed to define the action of the algebra $\mathcal{A}$ of observables on the state $|\psi_1\rangle \otimes |\psi_2\rangle$ of the system that is the composite system formed from two independent systems with wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$a \triangleright (|\psi_1\rangle \otimes |\psi_2\rangle) = \Delta(a)\,(|\psi_1\rangle \otimes |\psi_2\rangle) = a_{(1)}\,|\psi_1\rangle \otimes a_{(2)}\,|\psi_2\rangle \quad (\forall a \in \mathcal{A}). \tag{2.2.3}$$

In other words, for bialgebras it is possible to formulate a theory of representations in which new representations can be constructed by direct multiplication of old ones.

A classical example of a bialgebra is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, in particular, the spin algebra $\mathfrak{su}(2)$ in three-dimensional space. To demonstrate this, we consider the Lie algebra $\mathfrak{g}$ with generators $J_\alpha$ $(\alpha = 1, 2, 3, \dots)$, that satisfy the antisymmetric multiplication rule (defining relations)

$$[J_\alpha,\, J_\beta] = t^\gamma_{\alpha\beta} J_\gamma. \tag{2.2.4}$$

Here $t^\gamma_{\alpha\beta} = -t^\gamma_{\beta\alpha}$ are structure constants which satisfy Jacoby identity. The enveloping algebra of this algebra is the algebra $U(\mathfrak{g})$ with basis elements consisting of the identity $I$ and the elements $e_i = J_{\alpha_1} \cdots J_{\alpha_n}$ $\forall n \geqslant 1$, where the products of the generators $J$ are ordered lexicographically, i.e., $\alpha_1 \leqslant \alpha_2 \leqslant \dots \leqslant \alpha_n$. The coalgebraic structure for the algebra $U(\mathfrak{g})$ is specified by means of the mappings

$$\Delta(J_\alpha) = J_\alpha \otimes I + I \otimes J_\alpha, \quad \epsilon(J_\alpha) = 0, \quad \epsilon(I) = 1, \tag{2.2.5}$$

which satisfy all the axioms of a bialgebra. The mapping $\Delta$ in (2.2.5) is none other than the rule for addition of spins. In fact, one can quantize the coalgebraic structure (2.2.5) for universal enveloping algebra $U(\mathfrak{g})$ and consider the noncocommutative comultiplications $\Delta$. Such quantizations will be considered below in Subsection **3.3** and lead to the definition of Lie bialgebras.

Considering exponentials of elements of a Lie algebra, one can arrive at the definition of a group bialgebra of the group $G$ with structure mappings

$$\Delta(h) = h \otimes h, \quad \epsilon(h) = 1 \quad (\forall\, h \in G), \tag{2.2.6}$$

which obviously follow from (2.2.5). The next important example of a bialgebra is the algebra $\mathcal{A}(G)$ of functions $f$ on a group $(f : G \to \mathbb{C})$. This algebra is dual to the group algebra of the group $G$, and its structure mappings have the form $(f, f' \in \mathcal{A}(G);\;\; h, h' \in G)$:

$$(f \cdot f')(h) = f(h) f'(h), \quad f(h \cdot h') = (\Delta(f))(h, h') = f_{(1)}(h)\, f_{(2)}(h'), \quad \epsilon(f) = f(I), \tag{2.2.7}$$

where $I_G$ is the identity element in the group $G$. In particular, if the functions $T^i_j$ realize a matrix representation of the group $G$, then we have

$$T^i_j(hh') = T^i_k(h)T^k_j(h') \;\;\Rightarrow\;\; \Delta(T^i_j) = T^i_k \otimes T^k_j, \tag{2.2.8}$$

(the functions $T^i_j$ can be regarded as generators of a subalgebra in the algebra $\mathcal{A}(G)$). Note that if $\mathfrak{g}$ is non-Abelian, then $U(\mathfrak{g})$ and $G$ are noncommutative but cocommutative bialgebras, whereas $\mathcal{A}(G)$ is a commutative but noncocommutative bialgebra. Anticipating, we mention that the most interesting quantum groups are associated with noncommutative and noncocommutative bialgebras.

It is obvious that for a bialgebra $\mathcal{H}$ it is also possible to introduce the concepts of left (co)modules and (co)module (co)algebras (right (co)modules and (co)module (co)algebras are introduced in exactly the same way). Moreover, for the bialgebra $\mathcal{H}$ it is possible to introduce the concept of a left (right) bimodule $B$, i.e., a left (right) $\mathcal{H}$-module that is simultaneously a left (right) $\mathcal{H}$-comodule; at the same time, the module and comodule structures must be self-consistent:

$$\Delta_L(\mathcal{H} \triangleright B) = \Delta(\mathcal{H}) \triangleright \Delta_L(B),$$
$$(\epsilon \otimes id)\Delta_L(b) = b, \quad b \in B,$$

where $\Delta_L(b) = \bar{b}^{(-1)} \otimes b^{(0)}$ and $\bar{b}^{(-1)} \in \mathcal{H}$, $b^{(0)} \in B$. On the other hand, in the case of bialgebras, the conditions of conserving of the (co)algebraic structure of (co)modules can be represented in a more explicit form. For example, for the left $\mathcal{H}$-module algebra $\mathcal{A}$ we have ($a, b \in \mathcal{A}$; $h \in \mathcal{H}$):

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright I_A = \epsilon(h)I_A.$$

In addition, for the left $\mathcal{H}$-module coalgebra $\mathcal{A}$ we must have

$$\Delta(h \triangleright a) = \Delta(h) \triangleright \Delta(a) = (h_{(1)} \triangleright a_{(1)}) \otimes (h_{(2)} \triangleright a_{(2)}), \quad \epsilon(h \triangleright a) = \epsilon(h)\epsilon(a).$$

Similarly, the algebra $\mathcal{A}$ is a left $\mathcal{H}$-comodule algebra if

$$\Delta_L(ab) = \Delta_L(a)\,\Delta_L(b), \quad \Delta_L(I_A) = I_{\mathcal{H}} \otimes I_A,$$

and, finally, the coalgebra $\mathcal{A}$ is a left $\mathcal{H}$-comodule coalgebra if

$$(id \otimes \Delta)\Delta_L(a) = m_{\mathcal{H}}(\Delta_L \otimes \Delta_L)\Delta(a), \quad (id \otimes \epsilon_A)\Delta_L(a) = I_{\mathcal{H}}\epsilon_A(a), \tag{2.2.9}$$

where

$$m_{\mathcal{H}}(\Delta_L \otimes \Delta_L)(a \otimes b) = \bar{a}^{(-1)}\bar{b}^{(-1)} \otimes a^{(0)} \otimes b^{(0)}.$$

We now consider the bialgebra $\mathcal{H}$, which acts on a certain module algebra $\mathcal{A}$. One further important property of bialgebras is that we can define a new associative algebra $\mathcal{A}\sharp\mathcal{H}$ as the cross product (smash product) of $\mathcal{A}$ and $\mathcal{H}$. Namely:

**Definition 3.** *The left smash product $\mathcal{A}\sharp\mathcal{H}$ of the bialgebra $\mathcal{H}$ and its left module algebra $\mathcal{A}$ is an associative algebra such that:*
*1) as a vector space, $\mathcal{A}\sharp\mathcal{H}$ is identical to $\mathcal{A} \otimes \mathcal{H}$;*
*2) the product is defined in the sense ($h, g \in \mathcal{H}$; $a, b \in \mathcal{A}$):*

$$(a\sharp g)\,(b\sharp h) = \sum_g a(g_{(1)} \triangleright b)\sharp(g_{(2)}h) \equiv (a\sharp I)\,(\Delta(g) \triangleright (b\sharp h))\,; \tag{2.2.10}$$

*3) the identity element is $I\sharp I$.*

If the algebra $\mathcal{A}$ is the bialgebra dual to the bialgebra $\mathcal{H}$, then the relations (2.2.10) and (2.1.13) define the rules for interchanging the elements $(I\sharp g)$ and $(a\sharp I)$:

$$(I\sharp g)\,(a\sharp I) = (a_{(1)}\sharp I)\,\langle g_{(1)}|a_{(2)}\rangle\,(I\sharp g_{(2)}). \tag{2.2.11}$$

Thus, the subalgebras $\mathcal{A}$ and $\mathcal{H}$ in $\mathcal{A}\sharp\mathcal{H}$ do not commute with each other. The smash product depends on which action (left or right) of the algebra $\mathcal{H}$ on $\mathcal{A}$ we choose. In addition, the smash product generalizes the concept of the semidirect product. In particular, if we take as bialgebra $\mathcal{H}$ the Lorentz group algebra (see (2.2.6)), and as module $\mathcal{A}$ the group of translations in Minkowski space, then the smash product $\mathcal{A}\sharp\mathcal{H}$ defines the structure of the Poincare group.

The coanalog of the smash product, the smash coproduct $\mathcal{A}\underline{\sharp}\mathcal{H}$, can also be defined. For this, we consider the bialgebra $\mathcal{H}$ and its comodule coalgebra $\mathcal{A}$. Then on the space $\mathcal{A}\otimes\mathcal{H}$ it is possible to define the structure of a coassociative coalgebra

$$\Delta(a\,\underline{\sharp}\,h) = (a_{(1)}\,\underline{\sharp}\,\bar{a}_{(2)}^{(-1)}h_{(1)})\otimes(a_{(2)}^{(0)}\,\underline{\sharp}\,h_{(2)}), \quad \epsilon(a\,\underline{\sharp}\,h) = \epsilon(a)\epsilon(h). \tag{2.2.12}$$

The proof of the coassociativity reduces to verification of the identity

$$(m_{\mathcal{H}}(\Delta_L\otimes\Delta_{\mathcal{H}})\otimes id)(id\otimes\Delta_L)\Delta_{\mathcal{A}}(a) = (id\otimes id\otimes\Delta_L)(id\otimes\Delta_{\mathcal{A}})\Delta_L(a),$$

which is satisfied if we take into account the axiom (2.2.9) and the comodule axiom

$$(id\otimes\Delta_L)\Delta_L(a) = (\Delta_{\mathcal{H}}\otimes id)\Delta_L(a). \tag{2.2.13}$$

Note that from the two bialgebras $\mathcal{A}$ and $\mathcal{H}$, which act and coact on each other in a special manner, it is possible to organize a new bialgebra that is simultaneously the smash product and smash coproduct of $\mathcal{A}$ and $\mathcal{H}$ (bicross product; see [52]).

### *2.3. Hopf algebras. Universal $\mathcal{R}$-matrices*

We can now introduce the main concept in the theory of quantum groups, namely, the concept of the Hopf algebra.

**Definition 4.** *A bialgebra $\mathcal{A}$ equipped with an additional mapping $S:\ \mathcal{A}\to\mathcal{A}$ such that*

$$\begin{gathered} m(S\otimes id)\Delta = m(id\otimes S)\Delta = \mathbf{i}\cdot\epsilon \Rightarrow \\ S(a_{(1)})\,a_{(2)} = a_{(1)}\,S(a_{(2)}) = \epsilon(a)\cdot I \ \ (\forall a\in\mathcal{A}) \end{gathered} \tag{2.3.1}$$

*is called a Hopf algebra. The mapping $S$ is called the antipode and is an antihomomorphism with respect to both multiplication and comultiplication:*

$$S(ab) = S(b)S(a), \ \ (S\otimes S)\Delta(a) = \sigma\cdot\Delta(S(a)), \tag{2.3.2}$$

*where $a,b\in\mathcal{A}$ and $\sigma$ denotes the operator of transposition, $\sigma(a\otimes b) = (b\otimes a)$.*

If we set

$$S(e_i) = S_i^j e_j, \tag{2.3.3}$$

then the axiom (2.3.1) can be rewritten in the form

$$\Delta_k^{ij}S_i^n m_{nj}^l = \Delta_k^{ij}S_j^n m_{in}^l = \epsilon_k E^l. \tag{2.3.4}$$

From the axioms for the structure mappings of a Hopf algebra, it is possible to obtain the useful equations

$$S^i_j \epsilon_i = \epsilon_j, \quad S^i_j E^j = E^i,$$
$$\Delta^{ji}_k (S^{-1})^n_i m^l_{nj} = \Delta^{ji}_k (S^{-1})^n_j m^l_{in} = \epsilon_k E^l, \tag{2.3.5}$$

which we shall use in what follows. Note that, in general, the antipode $S$ is not necessarily invertible. An invertible antipode is called bijective.

In quantum physics, the existence of the antipode $S$ is needed to define a space of contragredient states $\langle\psi|$ (contragredient module of $\mathcal{A}$) with pairing $\langle\psi|\phi\rangle$: $\langle\psi| \otimes |\phi\rangle \to \mathbb{C}$. Left actions of the Hopf algebra $\mathcal{A}$ of observables to the contragredient states are (cf. the actions (2.2.3) of $\mathcal{A}$ to the states $|\psi_1\rangle \otimes |\psi_2\rangle$):

$$a \triangleright \langle\psi| := \langle\psi|\, S(a) \;\; (a \in \mathcal{A}), \tag{2.3.6}$$

$$a \triangleright (\langle\psi_1| \otimes \langle\psi_2|) := (\langle\psi_1| \otimes \langle\psi_2|)\Delta(S(a)) = \langle\psi_1|\, S(a_{(2)}) \otimes \langle\psi_2|\, S(a_{(1)}).$$

The states $\langle\psi|$ are called left dual to the states $|\phi\rangle$; the right dual ones are introduced with the help of the inverse antipode $S^{-1}$ (see, e.g., [60, 61]). Then the covariance of the pairing $\langle\psi|\phi\rangle$ under the left action of $\mathcal{A}$ can be established:

$$a \triangleright \langle\psi|\phi\rangle \equiv (a_{(1)} \triangleright \langle\psi|)\; (a_{(2)} \triangleright |\phi\rangle) = \langle\psi|S(a_{(1)})\, a_{(2)}|\phi\rangle = \epsilon(a)\langle\psi|\phi\rangle,$$
$$a \triangleright \langle\psi_1|\phi_1\rangle\; \langle\psi_2|\phi_2\rangle = a \triangleright (\langle\psi_1| \otimes \langle\psi_2|)(|\phi_1\rangle \otimes |\phi_2\rangle) =$$
$$= \langle\psi_1|S(a_{(2)})a_{(3)}|\phi_1\rangle\langle\psi_1|S(a_{(1)})a_{(4)}|\phi_1\rangle = \epsilon(a)\; \langle\psi_1|\phi_1\rangle\; \langle\psi_2|\phi_2\rangle.$$

The universal enveloping algebra $U(\mathfrak{g})$ and the group bialgebra of the group $G$ that we considered above can again serve as examples of cocommutative Hopf algebras. An example of a commutative (but noncocommutative) Hopf algebra is the bialgebra $\mathcal{A}(G)$, which we also considered above. The antipodes for these algebras have the form

$$\begin{array}{r} U(\mathfrak{g}): \;\; S(J_\alpha) = -J_\alpha, \quad S(I) = I, \\ G: \;\; S(h) = h^{-1}, \\ \mathcal{A}(G): \;\; S(f)(h) = f(h^{-1}), \end{array} \tag{2.3.7}$$

and satisfy the relation $S^2 = id$, which holds for all commutative or cocommutative Hopf algebras.

From the point of view of the axiom (2.3.1), $S(a)$ looks like the inverse of the element $a$, although in the general case $S^2 \neq id$. We recall that if a set $\mathcal{G}$ of elements with associative multiplication $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ and with identity (semigroup) also contains all the inverse elements, then such a set $\mathcal{G}$ becomes a group. Thus, from the point of view of the presence of the mapping $S$, a Hopf algebra generalizes the notion of the group algebra (for which $S(h) = h^{-1}$), although by itself it obviously does not need to be a group algebra. In accordance with Drinfeld's definition [13], the concepts of a Hopf algebra and a quantum group are more or less equivalent. Of course, the most interesting examples of quantum groups arise when one considers noncommutative and noncocommutative Hopf algebras.

Consider a noncommutative Hopf algebra $\mathcal{A}$ which is also noncocommutative $\Delta \neq \Delta^{\mathsf{op}} \equiv \sigma\Delta$, where $\sigma$ is the transposition operator $\sigma(a \otimes b) = b \otimes a$ $(\forall a, b \in \mathcal{A})$.

**Definition 5.** *A Hopf algebra $\mathcal{A}$ for which there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ such that*

$$\Delta^{\mathsf{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad \forall a \in \mathcal{A}, \tag{2.3.8}$$

$$(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.3.9}$$

*is called quasitriangular. Here the element*

$$\mathcal{R} = \sum_{ij} R^{(ij)} e_i \otimes e_j \tag{2.3.10}$$

*is called the universal $\mathcal{R}$-matrix, $R^{(ij)} \in \mathbb{C}$ are the constants and the symbols $\mathcal{R}_{12}, \mathcal{R}_{13}, \mathcal{R}_{23}$ have the meaning*

$$\mathcal{R}_{12} = \sum_{ij} R^{(ij)} e_i \otimes e_j \otimes I, \quad \mathcal{R}_{13} = \sum_{ij} R^{(ij)} e_i \otimes I \otimes e_j, \quad \mathcal{R}_{23} = \sum_{ij} R^{(ij)} I \otimes e_i \otimes e_j. \tag{2.3.11}$$

The relation (2.3.8) shows that the noncocommutativity in a quasitriangular Hopf algebra is kept "under control". It can be shown [51] that for such a Hopf algebra the universal $\mathcal{R}$-matrix (2.3.10) satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.3.12}$$

(to which a considerable part of the review will be devoted) and the relations

$$(id \otimes \epsilon)\mathcal{R} = (\epsilon \otimes id)\mathcal{R} = I, \tag{2.3.13}$$

$$\begin{aligned} (S \otimes id)\mathcal{R} = \mathcal{R}^{-1} \quad &\Leftrightarrow \quad (S^{-1} \otimes id)\mathcal{R}^{-1} = \mathcal{R}, \\ (id \otimes S)\mathcal{R}^{-1} = \mathcal{R} \quad &\Leftrightarrow \quad (id \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1}. \end{aligned} \tag{2.3.14}$$

The Yang–Baxter equation (2.3.12) follows from (2.3.8) and (2.3.9):

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{12}\,(\Delta \otimes id)(\mathcal{R}) = (\Delta^{\mathsf{op}} \otimes id)(\mathcal{R})\,\mathcal{R}_{12} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12}. \tag{2.3.15}$$

It is easy to derive the relations (2.3.13) by applying $(\epsilon \otimes id \otimes id)$ and $(id \otimes id \otimes \epsilon)$, respectively, to the first and second relation in (2.3.9), and then taking into account (2.1.9). Next, we prove the equalities in (2.3.14). We consider expressions $\mathcal{R} \cdot (S \otimes id)\,\mathcal{R}$ and $\mathcal{R} \cdot (id \otimes S^{-1})\,\mathcal{R}$ and make use of the Hopf algebra axioms (2.3.1) and equations (2.3.9) and (2.3.13):

$$\begin{gathered} \mathcal{R}_{23} \cdot (id \otimes S \otimes id)\,\mathcal{R}_{23} = (m_{12} \otimes id_3)\big(\mathcal{R}_{13}\,(id \otimes S \otimes id)\mathcal{R}_{23}\big) = \\ = (m_{12} \otimes id_3)\,(id \otimes S \otimes id)\mathcal{R}_{13}\,\mathcal{R}_{23} = (m_{12} \otimes id_3)\,\big((id \otimes S)\Delta \otimes id\big)\mathcal{R} = (\mathbf{i} \cdot \epsilon \otimes id)\,\mathcal{R} = I, \end{gathered}$$

$$\begin{gathered} \mathcal{R}_{12}\,(id \otimes S^{-1})\,\mathcal{R}_{12} = (id_1 \otimes m_{23})\,(id \otimes id \otimes S^{-1})\mathcal{R}_{12}\,\mathcal{R}_{13} = \\ = (id_1 \otimes m_{23})\,\big(id \otimes (id \otimes S^{-1})\Delta^{\mathsf{op}}\big)\mathcal{R} = (id \otimes \mathbf{i} \cdot \epsilon)\,\mathcal{R} = I, \end{gathered}$$

where the ultimate equality follows from (2.3.1) which is written in the form $a_{(2)} S^{-1}(a_{(1)}) = \epsilon(a) I$.

The next important concept that we shall need in what follows is the concept of the Hopf algebra $\mathcal{A}^*$ that is the dual of the Hopf algebra $\mathcal{A}$. We choose in $\mathcal{A}^*$ basis elements $\{e^i\}$ and define multiplication, identity, comultiplication, coidentity, and antipode for $\mathcal{A}^*$ in the form

$$e^i e^j = m^{ij}_k e^k, \;\; I = \bar{E}_i e^i, \;\; \Delta(e^i) = \Delta^i_{jk} e^j \otimes e^k, \;\; \epsilon(e^i) = \bar{\epsilon}^i, \;\; S(e^i) = \bar{S}^i_j e^j. \tag{2.3.16}$$

**Definition 6.** *Two Hopf algebras $\mathcal{A}$ and $\mathcal{A}^*$ with corresponding bases $\{e_i\}$ and $\{e^i\}$ are said to be dual to each other if there exists a nondegenerate pairing $\langle .|.\rangle$: $\mathcal{A}^* \otimes \mathcal{A} \to \mathbb{C}$ such that*

$$\langle e^i e^j | e_k \rangle \equiv \langle e^i \otimes e^j | \Delta(e_k)\rangle = \langle e^i | e_{k'}\rangle \Delta_k^{k'k''} \langle e^j | e_{k''}\rangle,$$

$$\langle e^i | e_j e_k \rangle \equiv \langle \Delta(e^i) | e_j \otimes e_k \rangle = \langle e^{i'} | e_j \rangle \Delta^i_{i'i''} \langle e^{i''} | e_k \rangle, \tag{2.3.17}$$

$$\langle S(e^i) | e_j \rangle = \langle e^i | S(e_j)\rangle, \quad \langle e^i | I \rangle = \epsilon(e^i), \ \ \langle I | e_i \rangle = \epsilon(e_i).$$

Since the pairing $\langle .|.\rangle$ (2.3.17) is nondegenerate, we can always choose basis elements $\{e^i\}$ such that

$$\langle e^i | e_j \rangle = \delta^i_j. \tag{2.3.18}$$

Then from the axioms for the pairing (2.3.17) and from the definitions of the structure maps (2.1.1), (2.3.3), and (2.3.16) in $\mathcal{A}$ and $\mathcal{A}^*$ we readily deduce

$$m_k^{ij} = \Delta_k^{ij}, \quad m_{ij}^k = \Delta_{ij}^k, \quad \bar{S}^i_j = S^i_j, \quad \bar{\epsilon}^i = E^i, \quad \bar{E}_i = \epsilon_i. \tag{2.3.19}$$

Thus, the multiplication, identity, comultiplication, coidentity, and antipode in a Hopf algebra define, respectively, comultiplication, coidentity, multiplication, identity, and antipode in the dual Hopf algebra.

**Remark.** In [63], L. Pontryagin showed that the set of characters of an Abelian locally compact group $G$ is an Abelian group, called the dual group $G^*$ of $G$. The group $G^*$ is also locally compact. Moreover, the dual group of $G^*$ is isomorphic to $G$. This beautiful theory becomes wrong if $G$ is a noncommutative group, even if it is finite. To restore the duality principle, one can replace the set of characters for a finite noncommutative group $G$ by the category of its irreducible representations (irreducible representations for the commutative groups are exactly characters). Indeed, T. Tannaka and M. Krein showed that the compact group $G$ can be recovered from the set of its irreducible unitary representations. They proved a duality theorem for compact groups, involving irreducible representations of $G$ (although no group-like structure is to be put on that class, since the tensor product of two irreducible representations may no longer be irreducible). However, the tensor product of two irreducible representations of the compact group $G$ can be expanded as a sum of irreducible representations and, thus, the dual object has the structure of an algebra. Recall (see (2.2.8)) that matrix representations of group $G$ are realized by the sets of special functions $T^i_{\ k}$. One can consider the group algebra $\mathcal{G}$ of finite group $G$ and the algebra $\mathcal{A}(G) \equiv \mathcal{G}^\star$ of functions on the group $G$ as simplest examples of the Hopf algebras. The structure mappings for these algebras have been defined in (2.2.6), (2.2.7), and (2.3.7). Note that the algebras $\mathcal{G}$ and $\mathcal{G}^\star$ are Hopf dual to each other. The detailed structure of $\mathcal{G}^\star$ follows from the representation theory of finite groups (see, e.g., [54]).

### *2.4. Heisenberg and quantum doubles. Yetter–Drinfeld modules*

In Subsection **2.2**, we have defined (see Definition 3) the notion of the smash (cross) product of the bialgebra and its module algebra. Since the Hopf dual algebra $\mathcal{A}^*$ is the natural right and left module algebra for the Hopf algebra $\mathcal{A}$ (2.1.12), (2.1.13), one can immediately define the right $\mathcal{A}^* \sharp \mathcal{A}$ and the left $\mathcal{A} \sharp \mathcal{A}^*$ cross products of the algebra $\mathcal{A}$ on $\mathcal{A}^*$. These cross-product algebras are called Heisenberg doubles of $\mathcal{A}$ and they are the associative algebras with nontrivial cross-multiplication rules (cf. Eq. (2.2.11)):

$$a\, \bar{a} = (a_{(1)} \triangleright \bar{a})\, a_{(2)} = \bar{a}_{(1)}\, \langle a_{(1)} \,|\, \bar{a}_{(2)} \rangle\, a_{(2)}, \tag{2.4.1}$$

$$\bar{a}\, a = a_{(1)}(\bar{a} \triangleleft a_{(2)}) \; = a_{(1)} \, \langle \bar{a}_{(1)} \, | \, a_{(2)} \rangle \, \bar{a}_{(2)}, \tag{2.4.2}$$

where $a \in \mathcal{A}$ and $\bar{a} \in \mathcal{A}^*$. Here we discuss only the left cross-product algebra $\mathcal{A}\sharp\mathcal{A}^*$ (2.4.1) (the other one (2.4.2) is considered analogously).

As in the previous subsection, we denote $\{e^i\}$ and $\{e_i\}$ as the dual basis elements of $\mathcal{A}^*$ and $\mathcal{A}$, respectively. In terms of this basis, we rewrite (2.4.1) in the form

$$e_r \, e^n = e^i \, \Delta^n_{if} \langle e_j | e^f \rangle \, \Delta^{jk}_r \, e_k = m^n_{ij} \, e^i \, e_k \, \Delta^{jk}_r. \tag{2.4.3}$$

Let us define a right $\mathcal{A}^*$-coaction and a left $\mathcal{A}$-coaction on the algebra $\mathcal{A}\sharp\mathcal{A}^*$ such that these coactions respect the algebra structure of $\mathcal{A}\sharp\mathcal{A}^*$:

$$\Delta_R(z) = C \, (z \otimes 1) \, C^{-1}, \quad \Delta_L(z) = C^{-1} \, (1 \otimes z) \, C, \quad C \equiv e_i \otimes e^i. \tag{2.4.4}$$

The inverse of the canonical element $C$ is

$$C^{-1} = S(e_i) \otimes e^i = e_i \otimes S(e^i),$$

and $\Delta_R$, $\Delta_L$ (2.4.4) are represented in the form

$$\Delta_R(z) = (e_{k(1)} \, z \, S(e_{k(2)})) \otimes e^k, \quad \Delta_L(z) = e_k \otimes S(e^k_{(1)}) \, z \, e^k_{(2)}. \tag{2.4.5}$$

Note that $\Delta_R(\bar{z}) = \Delta(\bar{z}) \; \forall \bar{z} \in \mathcal{A}^*$ and $\Delta_L(z) = \Delta(z) \; \forall z \in \mathcal{A}$ (here $\mathcal{A}$ and $\mathcal{A}^*$ are understood as the Hopf subalgebras in $\mathcal{A}\sharp\mathcal{A}^*$ and $\Delta$ are corresponding comultiplications). Indeed, for $z \in \mathcal{A}$ we have

$$\Delta_L(z) = e_k \otimes S(e^k_{(1)}) \, z \, e^k_{(2)} = e_k \otimes S(e^k_{(1)}) \, e^k_{(2)} \, \langle z_{(1)} \, | \, e^k_{(3)} \rangle z_{(2)} =$$

$$= e_k \, \langle z_{(1)} \, | \, e^k \rangle \otimes z_{(2)} = z_{(1)} \otimes z_{(2)},$$

(the proof of $\Delta_R(\bar{z}) = \Delta(\bar{z})$ is similar). The axioms

$$(id \otimes \Delta)\Delta_R = (\Delta_R \otimes id)\Delta_R, \quad (id \otimes \Delta_L)\Delta_L = (\Delta \otimes id)\Delta_L,$$

$$(id \otimes \Delta_R)\Delta_L(z) = C^{-1}_{13} \, (\Delta_L \otimes id)\Delta_R(z) \, C_{13}$$

can be verified directly by using relations (cf. (2.3.9))

$$(id \otimes \Delta)C_{12} = C_{13} \, C_{23}, \quad (\Delta \otimes id)C_{12} = C_{13} \, C_{23}$$

and the pentagon identity [55, 56] for $C$:

$$C_{12} \, C_{13} \, C_{23} = C_{23} \, C_{12}. \tag{2.4.6}$$

The proof of (2.4.6) is straightforward (see (2.4.3)):

$$C_{12} \, C_{13} \, C_{23} = e_i \, e_j \otimes e^i \, e_k \otimes e^j \, e^k = e_n \otimes m^n_{ij} \, e^i \, e_k \, \Delta^{jk}_r \otimes e^r =$$

$$= e_n \otimes e_r \, e^n \otimes e^r = C_{23} \, C_{12}.$$

The pentagon identity (2.4.6) is used for the construction of the explicit solutions of the tetrahedron equations (3D generalizations of Yang–Baxter equations).

Although $\mathcal{A}$ and $\mathcal{A}^*$ are Hopf algebras, their Heisenberg doubles $\mathcal{A}\sharp\mathcal{A}^*$, $\mathcal{A}^*\sharp\mathcal{A}$ are not Hopf algebras. But as we have just seen before, the algebra $\mathcal{A}\sharp\mathcal{A}^*$ (as well as $\mathcal{A}^*\sharp\mathcal{A}$) still possesses some covariance properties, since the coactions (2.4.4) are covariant transformations (homomorphisms) of the algebra $\mathcal{A}\sharp\mathcal{A}^*$.

The natural question is the following: is it possible to invent such a cross-product of the Hopf algebra and its dual Hopf algebra to obtain a new Hopf algebra? V. Drinfeld [10] showed that there exists a quasitriangular Hopf algebra $\mathcal{D}(\mathcal{A})$ that is a special smash product of the Hopf algebras $\mathcal{A}$ and $\mathcal{A}^o$: $\mathcal{D}(\mathcal{A}) = \mathcal{A} \bowtie \mathcal{A}^o$, which is called the quantum double. Here we denote by $\mathcal{A}^o$ the algebra $\mathcal{A}^*$ with opposite comultiplication: $\Delta(e^i) = m^i_{kj} e^j \otimes e^k$, $\mathcal{A}^o = (\mathcal{A}^*)^{\rm cop}$. It follows from (2.3.5) that the antipode for $\mathcal{A}^o$ will be not $S$ but the skew antipode $S^{-1}$. Thus, the structure mappings for $\mathcal{A}^o$ have the form

$$e^i e^j = \Delta^{ij}_k e^k, \quad \Delta(e^i) = m^i_{kj} e^j \otimes e^k, \;\; S(e^i) = (S^{-1})^i_j e^j. \tag{2.4.7}$$

The algebras $\mathcal{A}$ and $\mathcal{A}^o$ are said to be antidual, and for them we can introduce the antidual pairing $\langle\langle . | . \rangle\rangle$: $\mathcal{A}^o \otimes \mathcal{A} \to \mathbb{C}$, which satisfies the conditions

$$\begin{gathered}
\langle\langle e^i e^j | e_k \rangle\rangle \equiv \langle\langle e^i \otimes e^j | \Delta(e_k) \rangle\rangle = \Delta^{ij}_k , \\
\langle\langle e^i | e_k e_j \rangle\rangle \equiv \langle\langle \Delta(e^i) | e_j \otimes e_k \rangle\rangle = m^i_{kj} , \\
\langle\langle S(e^i) | e_j \rangle\rangle = \langle\langle e^i | S^{-1}(e_j) \rangle\rangle = (S^{-1})^i_j , \\
\langle\langle e^i | S(e_j) \rangle\rangle = \langle\langle S^{-1}(e^i) | e_j \rangle\rangle = S^i_j , \\
\langle\langle e^i | I \rangle\rangle = E^i , \quad \langle\langle I | e_i \rangle\rangle = \epsilon_i .
\end{gathered} \tag{2.4.8}$$

The universal $R$-matrix can be expressed in the form of the canonical element

$$\mathcal{R} = (e_i \bowtie I) \otimes (I \bowtie e^i), \tag{2.4.9}$$

and the multiplication in $\mathcal{D}(\mathcal{A})$ is defined in accordance with (the summation signs are omitted)

$$(a \bowtie \alpha)(b \bowtie \beta) = a\left((\alpha_{(3)} \triangleright b) \triangleleft S(\alpha_{(1)})\right) \bowtie \alpha_{(2)}\beta, \tag{2.4.10}$$

where $\alpha, \beta \in \mathcal{A}^o$, $a, b \in \mathcal{A}$, $\Delta^2(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}$ and

$$\alpha \triangleright b = b_{(1)} \langle\langle \alpha | b_{(2)} \rangle\rangle, \;\; b \triangleleft \alpha = \langle\langle \alpha | b_{(1)} \rangle\rangle b_{(2)}. \tag{2.4.11}$$

The coalgebraic structure on the quantum double is defined by the direct product of the coalgebraic structures on the Hopf algebras $\mathcal{A}$ and $\mathcal{A}^o$:

$$\Delta(e_i \bowtie e^j) = \Delta(e_i \bowtie I)\Delta(I \bowtie e^j) = \Delta^{nk}_i m^j_{lp} (e_n \bowtie e^p) \otimes (e_k \bowtie e^l). \tag{2.4.12}$$

Finally, the antipode and coidentity for $\mathcal{D}(\mathcal{A})$ have the form

$$S(a \bowtie \alpha) = S(a) \bowtie S(\alpha), \quad \epsilon(a \bowtie \alpha) = \epsilon(a)\epsilon(\alpha). \tag{2.4.13}$$

All the axioms of a Hopf algebra can be verified for $\mathcal{D}(\mathcal{A})$ by direct calculation. A simple proof of the associativity of the multiplication (2.4.10) and the coassociativity of the comultiplication (2.4.12) can be found in [53].

Taking into account (2.4.11), we can rewrite (2.4.10) as the commutator for the elements $(I \bowtie \alpha)$ and $(b \bowtie I)$:

$$(I \bowtie \alpha)(b \bowtie I) = \langle\langle S(\alpha_{(1)})|b_{(1)}\rangle\rangle(b_{(2)} \bowtie I)(I \bowtie \alpha_{(2)})\langle\langle \alpha_{(3)}|b_{(3)}\rangle\rangle \tag{2.4.14}$$

or, in terms of the basis elements $\alpha = e^t$ and $b = e_s$, we have [10]:

$$\begin{gathered}(I \bowtie e^t)(e_s \bowtie I) = m^t_{klp}\Delta^{njk}_s (S^{-1})^p_n (e_j \bowtie I)(I \bowtie e^l) \equiv \\ \equiv \left(m^t_{ip}(S^{-1})^p_n \Delta^{nr}_s\right)\left(m^i_{kl}\Delta^{jk}_r\right)(e_j \bowtie I)(I \bowtie e^l),\end{gathered} \tag{2.4.15}$$

where $m^t_{klp}$ and $\Delta^{njk}_s$ are defined in (2.1.3) and (2.1.8), and $(S^{-1})^p_n$ is the matrix of the skew antipode.

The consistence of definitions of left and right bimodules over the quantum double $\mathcal{D}(\mathcal{A})$ should be clarified in view of the nontrivial structure of the cross-multiplication rule (2.4.14), (2.4.15) for subalgebras $\mathcal{A}$ and $\mathcal{A}^o$. It can be done (see, e.g., [198]) if one considers left or right coinvariant bimodules (Hopf modules): $M^L = \{m : \ \Delta_L(m) = 1 \otimes m\}$ or $M^R = \{m : \Delta_R(m) = m \otimes 1\}$. For example, for $M^R$ one can define the left $\mathcal{A}$ and left $\mathcal{A}^o$-module actions as

$$a \triangleright m = a_{(1)}\, m\, S(a_{(2)}), \tag{2.4.16}$$

$$\alpha \triangleright m = \langle\langle S(\alpha), m_{(-1)}\rangle\rangle\, m_{(0)}, \tag{2.4.17}$$

where $\Delta_L(m) = m_{(-1)} \otimes m_{(0)}$ is the left $\mathcal{A}$-coaction on $M^R$ and $a \in \mathcal{A}$, $\alpha \in \mathcal{A}^o$. Note that the left $\mathcal{A}$-module action (2.4.16) respects the right coinvariance of $M^R$. The compatibility condition for the left $\mathcal{A}$-action (2.4.16) and the left $\mathcal{A}$-coaction $\Delta_L$ is written in the form (we represent $\Delta_L(a \triangleright m)$ in two different ways):

$$(a \triangleright m)_{(-1)} \otimes (a \triangleright m)_{(0)} = a_{(1)}\, m_{(-1)}\, S(a_{(3)}) \otimes a_{(2)} \triangleright m_{(0)}. \tag{2.4.18}$$

A module with the property (2.4.18) is called the Yetter–Drinfeld module. Then, using (2.4.16), (2.4.18) and opposite coproduct for $\mathcal{A}^o$, we obtain

$$\begin{gathered}\alpha \triangleright (a \triangleright m) = \alpha \triangleright (a_{(1)}\, m\, S(a_{(2)})) = \langle\langle S(\alpha), a_{(1)}\, m_{(-1)}\, S(a_{(3)})\rangle\rangle\, a_{(2)} \triangleright m_{(0)} = \\ = \langle\langle S(\alpha_{(1)}),\, a_{(1)}\rangle\rangle\, \langle\langle \alpha_{(3)},\, a_{(3)}\rangle\rangle\, a_{(2)} \triangleright (\alpha_{(2)} \triangleright m),\end{gathered} \tag{2.4.19}$$

and one can recognize in Eq. (2.4.19) the quantum double multiplication formula (2.4.14).

It follows from Eqs. (2.1.3), (2.1.8) and from the identities for the skew antipode (2.3.5) that

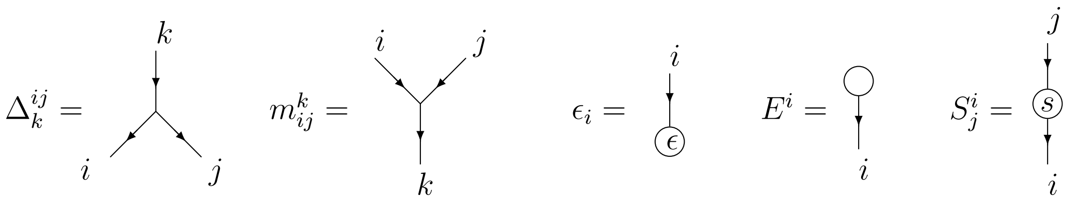
$$\left(m^q_{tk}\Delta^{ks}_m\right)\left(m^t_{ip}(S^{-1})^p_n \Delta^{nr}_s\right) = \delta^q_i \delta^r_m, \tag{2.4.20}$$

and this enables us to rewrite (2.4.15) in the form

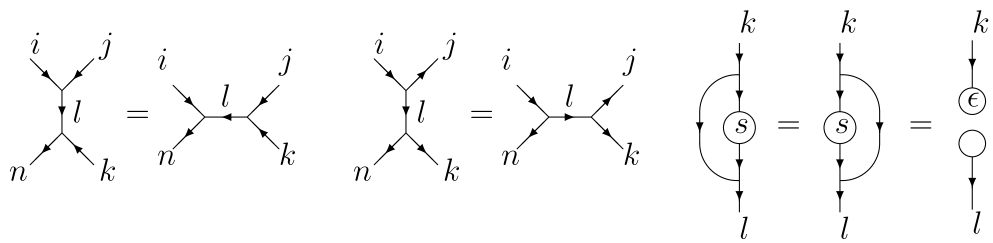
$$\left(m^q_{tk}\Delta^{ks}_m\right)(I \bowtie e^t)(e_s \bowtie I) = (m^q_{kl}\Delta^{jk}_m)(e_j \bowtie I)(I \bowtie e^l).$$

This equation is equivalent to the axiom (2.3.8) for the universal matrix $\mathcal{R}$ (2.4.9). The relations (2.3.9) for $\mathcal{R}$ (2.4.9) are readily verified. Thus, $\mathcal{D}(\mathcal{A})$ is indeed a quasitriangular Hopf algebra with universal $\mathcal{R}$-matrix represented by (2.4.9).

In conclusion, we note that many relations for the structure constants of Hopf algebras (for example, the relation (2.4.5)) can be obtained and represented in a transparent form by means of the following diagrammatic technique:



For example, the axioms of associativity (2.3) and coassociativity (2.1.8) and the axioms for the antipode (2.3.4) can be represented in the form



Now we make three important remarks relating to the further development of the theory of Hopf algebras.

*2.5. Twisted, ribbon and quasi-Hopf algebras*

**Remark 1.** *Twisted Hopf algebras.*
Consider a Hopf algebra $\mathcal{A}$ $(\Delta,\, \epsilon,\, S)$. Let $\mathcal{F}$ be an invertible element of $\mathcal{A} \otimes \mathcal{A}$ such that

$$(\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}\ , \tag{2.5.1}$$

and we denote $\mathcal{F} = \sum_i \alpha_i \otimes \beta_i$, $\mathcal{F}^{-1} = \sum_i \gamma_i \otimes \delta_i$, $I \equiv 1$. Following the twisting procedure [60, 61], one can define a new Hopf algebra $\mathcal{A}^{(F)}$ $(\Delta^{(F)},\, \epsilon^{(F)},\, S^{(F)})$ (twisted Hopf algebra) with the new structure mappings

$$\Delta^{(F)}(a) = \mathcal{F}\, \Delta(a)\, \mathcal{F}^{-1}, \tag{2.5.2}$$

$$\epsilon^{(F)}(a) = \epsilon(a), \quad S^{(F)}(a) = U\, S(a)\, U^{-1} (\forall a \in \mathcal{A}), \tag{2.5.3}$$

where the twisting element $\mathcal{F}$ satisfies the cocycle equation

$$\mathcal{F}_{12}\, (\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}\, (id \otimes \Delta)\mathcal{F}, \tag{2.5.4}$$

and the element $U = \alpha_i\, S(\beta_i)$ is invertible and obeys

$$U^{-1} = S(\gamma_i)\, \delta_i, \quad S(\alpha_i)\, U^{-1}\, \beta_i = 1 \tag{2.5.5}$$

(the summation over $i$ is assumed). First of all, we show that the algebra $\mathcal{A}^{(F)}$ $(\Delta^{(F)},\, \epsilon)$ is a bialgebra. Indeed, the cocycle equation (2.5.4) guarantees the coassociativity condition (2.1.8) for the new coproduct $\Delta^{(F)}$ (2.5.2). Then the axioms for counit $\epsilon$ (2.1.9) are easily deduced from (2.5.1). Considering the identity

$$m(id \otimes S \otimes id)\left(\mathcal{F}_{23}^{-1}\, \mathcal{F}_{12}\, (\Delta \otimes id)\mathcal{F}\right) = m(id \otimes S \otimes id)(id \otimes \Delta)\mathcal{F},$$

we obtain the form for $U^{-1}$ (2.5.5). The second relation in (2.5.5) is obtained from the identity $m(S \otimes id)\mathcal{F}^{-1}\mathcal{F} = 1$.

Now the new antipode $S^{(F)}$ (2.5.3) follows from equation

$$m(id \otimes S)\,(\Delta^{(F)}(a)\,\mathcal{F}) = m(id \otimes S)\,(\mathcal{F}\,\Delta(a)),$$

which is rewritten in the form $\tilde{a}_{(1)} U S(\tilde{a}_{(2)}) = \epsilon(a)\,U$, where $\Delta^{(F)}(a) = \tilde{a}_{(1)} \otimes \tilde{a}_{(2)}$.

If the algebra $\mathcal{A}$ is a quasitriangular Hopf algebra with the universal $\mathcal{R}$-matrix (2.3.8), then the new Hopf algebra $\mathcal{A}^{(F)}$ is also quasitriangular and a new universal $\mathcal{R}$-matrix is

$$\mathcal{R}^{(F)} = \mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}, \tag{2.5.6}$$

since we have

$$\Delta^{(F)'} = \mathcal{F}_{21}\,\Delta^{\mathsf{op}}\,\mathcal{F}_{21}^{-1} = \mathcal{F}_{21}\,\mathcal{R}\,\Delta\,\mathcal{R}^{-1}\,\mathcal{F}_{21}^{-1} = \left(\mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}\right)\Delta^{(F)}\left(\mathcal{F}\,\mathcal{R}^{-1}\,\mathcal{F}_{21}^{-1}\right).$$

The Yang–Baxter equation (2.3.12) for $R$-matrix (2.5.6) can be directly checked with the help of (2.3.8) and (2.5.4).

Impose additional relations on $\mathcal{F}$:

$$(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{13}\,\mathcal{F}_{23}, \quad (id \otimes \Delta)\mathcal{F} = \mathcal{F}_{13}\,\mathcal{F}_{12}, \tag{2.5.7}$$

which, together with (2.5.4), imply the Yang–Baxter equation for $\mathcal{F}$. Using (2.3.8), one deduces from (2.5.7) the equations

$$\mathcal{R}_{12}\,\mathcal{F}_{13}\,\mathcal{F}_{23} = \mathcal{F}_{23}\,\mathcal{F}_{13}\,\mathcal{R}_{12}, \quad \mathcal{F}_{12}\,\mathcal{F}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{F}_{13}\,\mathcal{F}_{12}. \tag{2.5.8}$$

Equations (2.5.8) and the Yang–Baxter relations for universal elements $\mathcal{R}$, $\mathcal{F}$ define the twist which is proposed in [57] (the additional condition $\mathcal{F}^{21}\mathcal{F} = 1 \otimes 1$ is assumed in [57]).

Note that if $\mathcal{A}$ is the Hopf algebra of functions on the group algebra of group $G$ (2.2.7), then Eq. (2.5.4) can be written in the form of 2-cocycle equation

$$\mathcal{F}(a,b)\,\mathcal{F}(ab,c) = \mathcal{F}(b,c)\mathcal{F}(a,bc), \quad (\forall a,b,c \in G),$$

for the projective representation $\rho$ of $G$: $\rho(a)\rho(b) = \mathcal{F}(a,b)\,\rho(a\,b)$. That is why Eq. (2.5.4) is called the cocycle equation.

Many explicit solutions of the cocycle equation (2.5.4) are known (see, e.g., [64–66] and references therein).

**Remark 2.** *Ribbon Hopf algebras.*
Here we explain the notion of the ribbon Hopf algebras [58]. Consider quasitriangular Hopf algebra $\mathcal{A}$ and represent the universal $\mathcal{R}$-matrix in the form

$$\mathcal{R} = \sum_{\mu} \alpha_\mu \otimes \beta_\mu, \quad \mathcal{R}^{-1} = \sum_{\mu} \gamma_\mu \otimes \delta_\mu, \tag{2.5.9}$$

where $\alpha_\mu$, $\beta_\mu$, $\gamma_\mu$, $\delta_\mu \in \mathcal{A}$. By using the right equalities in (2.3.14), we represent the identities $(id \otimes S)\big(\mathcal{R}\mathcal{R}^{-1}\big) = I = (id \otimes S)\big(\mathcal{R}^{-1}\mathcal{R}\big)$ as

$$\alpha_\mu\,\alpha_\nu \otimes \beta_\nu\,S(\beta_\mu) = I = \alpha_\mu\,\alpha_\nu \otimes S(\beta_\nu)\,\beta_\mu \tag{2.5.10}$$

(the summation over repeated indices $\mu$ and $\nu$ is assumed and we write $I$ instead of $(I \otimes I)$), while for $(S \otimes id)\mathcal{R}\mathcal{R}^{-1} = I = (S \otimes id)\mathcal{R}^{-1}\mathcal{R}$ we have

$$S(\gamma_\mu)\,\gamma_\nu \otimes \delta_\nu\,\delta_\mu = I = \gamma_\mu\,S(\gamma_\nu) \otimes \delta_\nu\,\delta_\mu. \tag{2.5.11}$$

We use identities (2.5.10) and (2.5.11) below in Subsection **3.1.2** (Remark 1).

Consider the element $u = \sum_\mu S(\beta_\mu)\,\alpha_\mu$ for which the following proposition holds.

**Proposition 2.1** *(see [51]).*
*1. For any $a \in \mathcal{A}$ we have*

$$S^2(a)\, u = u\, a. \tag{2.5.12}$$

*2. The element $u$ is invertible, with*

$$u^{-1} = S^{-1}(\delta_\mu)\, \gamma_\mu. \tag{2.5.13}$$

**Proof.** 1. From the relation (2.3.8) it follows that $\forall a \in \mathcal{A}$ (the summation signs are omitted):

$$\alpha_\mu\, a_{(1)} \otimes \beta_\mu\, a_{(2)} \otimes a_{(3)} = a_{(2)}\, \alpha_\mu \otimes a_{(1)}\, \beta_\mu \otimes a_{(3)},$$

where $a_{(1)} \otimes a_{(2)} \otimes a_{(3)} = (\Delta \otimes id)\Delta(a)$. From this we obtain

$$S^2(a_{(3)})\, S(\beta_\mu\, a_{(2)})\, \alpha_\mu a_{(1)} = S^2(a_{(3)})\, S(a_{(1)}\, \beta_\mu)\, a_{(2)}\alpha_\mu,$$

or

$$S^2(a_{(3)})\, S(a_{(2)})\, u\, a_{(1)} = S^2(a_{(3)})\, S(\beta_\mu)\, S(a_{(1)})\, a_{(2)}\alpha_\mu.$$

Applying to this equation the axioms (2.3.1), we obtain (2.5.12).
2. Putting $w = S^{-1}(\delta_\mu)\, \gamma_\mu$, we have

$$u\, w = u\, S^{-1}(\delta_\mu)\, \gamma_\mu = S(\delta_\mu)\, u\, \gamma_\mu = S(\beta_\nu\, \delta_\mu)\, \alpha_\nu\, \gamma_\mu.$$

Since $\mathcal{R} \cdot \mathcal{R}^{-1} = \alpha_\nu\gamma_\mu \otimes \beta_\nu\delta_\mu = I$, we have $u\, w = I$. It follows from the last equation and from (2.5.12) that $S^2(w)\, u = 1$, and therefore the element $u$ has both a right and left inverse (2.5.13). ∎
Thus, the element $u$ is invertible and we can rewrite (2.5.12) in the form

$$S^2(a) = u\, a\, u^{-1}. \tag{2.5.14}$$

This relation shows, in particular, that the operation of taking the antipode is not involutive.

**Proposition 2.2** *(see [51]).*
*Define the following elements:*

$$u_1 \equiv u = S(\beta_\mu)\, \alpha_\mu, \quad u_2 = S(\gamma_\mu)\, \delta_\mu, \quad u_3 = \beta_\mu\, S^{-1}(\alpha_\mu), \quad u_4 = \gamma_\mu\, S^{-1}(\delta_\mu). \tag{2.5.15}$$

*The relations (2.5.14) are satisfied if we take any of the elements $u_i$ from (2.5.15):*

$$S^2(a) = u_i\, a\, u_i^{-1}, \quad \forall a \in \mathcal{A}. \tag{2.5.16}$$

*In addition, we have $S(u_1)^{-1} = u_2$, $S(u_3)^{-1} = u_4$, and it turns out that all $u_i$ commute with each other, while the elements $u_i u_j^{-1} = u_j^{-1} u_i$ are central in $\mathcal{A}$. Consequently, the element $u\, S(u) = u_1\, u_2^{-1}$ is also central.*

**Proof.** In view of relation (2.5.13) we have $S(u_1)^{-1} = S(u^{-1}) = S(\gamma_\mu)\delta_\mu = u_2$ and $u_2^{-1} = S(u) = S^{-1}(u) = S^{-1}(\alpha_\mu)\, \beta_\mu$, where we use the identity $S^2(u) = u$ which follows from (2.5.14). Applying the map $S$ to both parts of (2.5.14), we deduce $S^3(a) = u_2 S(a) u_2^{-1}$ which is equivalent to (2.5.16) for $i = 2$. Note that from (2.3.14) we have $\mathcal{R}^{\pm} = (S^{-1} \otimes S^{-1})\mathcal{R}^{\pm}$. Thus, one can make in all formulas above the substitution $\alpha_\mu \to S^{-1}(\alpha_\mu)$, $\beta_\mu \to S^{-1}(\beta_\mu)$ and $\gamma_\mu \to S^{-1}(\gamma_\mu)$, $\delta_\mu \to S^{-1}(\delta_\mu)$ to exchange the elements $u_1$ and $u_2$, respectively, to the elements $u_3$ and $u_4$. It means that equations (2.5.16) are valid for $i = 3, 4$ and we have $u_4 = S(u_3)^{-1}$. Relations (2.5.16) yield $S^2(u_j) = u_j$ $(\forall j)$ and substitution $a = u_j$ to (2.5.16) gives $u_i u_j = u_j u_i$ $(\forall i, j = 1, \dots, 4)$. Finally, for any $a \in \mathcal{A}$ we have $u_j^{-1} u_i \, a \, u_i^{-1} u_j = u_j^{-1} \, S^2(a) \, u_j = a$, which means that elements $u_j^{-1} u_i = u_i u_j^{-1}$ are central. ■

In [51], it was noted that

$$\Delta(u) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(u \otimes u) = (u \otimes u)(\mathcal{R}_{21}\mathcal{R}_{12})^{-1}.$$

On the basis of all these propositions, we introduce the important concept of a ribbon Hopf algebra (see [58]):

**Definition 7.** *Consider a quasitriangular Hopf algebra* $(\mathcal{A}, \mathcal{R})$. *Then the triplet* $(\mathcal{A}, \mathcal{R}, v)$ *is called a ribbon Hopf algebra if* $v$ *is a central element in* $\mathcal{A}$ *and*

$$v^2 = u\, S(u), \quad S(v) = v, \quad \epsilon(v) = 1,$$

$$\Delta(v) = (\mathcal{R}_{21}\, \mathcal{R}_{12})^{-1}\, (v \otimes v).$$

For each quasitriangular Hopf algebra $\mathcal{A}$ we can define $\mathcal{A}$-colored ribbon graphs [58]. If, moreover, $\mathcal{A}$ is a ribbon Hopf algebra, then for each $\mathcal{A}$-colored ribbon graph we can associate the central element of $\mathcal{A}$ that generalizes the Jones polynomial being an invariant of a knot in $\mathbb{R}^3$ (see [58, 67]).

**Remark 3.** *Quasi-Hopf algebras.*

One can introduce a generalization of a Hopf algebra, called a quasi-Hopf algebra [60, 61], which is defined as an associative unital algebra $\mathcal{A}$ with homomorphism $\Delta : \ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, homomorphism $\epsilon : \ \mathcal{A} \to \mathbb{C}$, antiautomorphism $S : \ \mathcal{A} \to \mathcal{A}$, and invertible element $\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. At the same time, $\Delta$, $\epsilon$, $\Phi$, and $S$ satisfy the axioms

$$(id \otimes \Delta)\Delta(a) = \Phi \cdot (\Delta \otimes id)\Delta(a) \cdot \Phi^{-1}, \quad a \in \mathcal{A}, \tag{2.5.17}$$

$$(id \otimes id \otimes \Delta)(\Phi) \cdot (\Delta \otimes id \otimes id)(\Phi) = (I \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes I), \tag{2.5.18}$$

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta, \quad (id \otimes \epsilon \otimes id)\Phi = I \otimes I, \tag{2.5.19}$$

$$S(a_{(1)})\, \alpha\, a_{(2)} = \epsilon(a)\, \alpha, \quad a_{(1)}\, \beta\, S(a_{(2)}) = \epsilon(a)\, \beta, \tag{2.5.20}$$

$$\phi_i\, \beta\, S(\phi_i')\, \alpha\, \phi_i'' = I, \quad S(\bar{\phi}_i)\, \alpha\, \bar{\phi}_i'\, \beta\, S(\bar{\phi}_i'') = I,$$

where $\alpha$ and $\beta$ are certain fixed elements of $\mathcal{A}$, $\Delta(a) = a_{(1)} \otimes a_{(2)}$, and

$$\Phi := \phi_i \otimes \phi_i' \otimes \phi_i'', \quad \Phi^{-1} := \bar{\phi}_i \otimes \bar{\phi}_i' \otimes \bar{\phi}_i''$$

(summation over $i$ is assumed). Thus, a quasi-Hopf algebra differs from an ordinary Hopf algebra in that the axiom of coassociativity is replaced by the weaker condition (2.5.17). In other words, a quasi-Hopf algebra is noncoassociative, but this noncoassociativity is kept under control by means of the element $\Phi$. The axioms (2.5.20) (which look like different definitions of the left and right antipodes) generalize the axioms (2.3.1) for usual Hopf algebras and consequently the elements $\alpha$ and $\beta$ involved into the play with the contragredient representations of the quasi-Hopf algebras.

To make the pentagonal condition (2.5.18) more transparent, let us consider (following [60, 61]) the algebra $\mathcal{A}$ as the algebra of functions on a "noncommutative" space $X$ equipped with a $*$ product: $X \times X \to X$. Then elements $a \in \mathcal{A}$, $b \in \mathcal{A} \otimes \mathcal{A}, \dots$ are written in the form $a(x)$, $b(x,y)$ $\dots$ and $\Delta(a)$ is represented as $a(x * y)$. The homomorphism $\epsilon$ defines the point in $X$, which we denote 1 and instead of $\epsilon(a)$ we write $a(1)$. Then Eqs. (2.5.17)–(2.5.19) are represented in the form [60, 61]:

$$\begin{gathered} a(x*(y*z)) = \Phi(x,y,z)\, a((x*y)*z)\, \Phi(x,y,z)^{-1}, \\ \Phi(x,y,z*u)\, \Phi(x*y,z,u) = \Phi(y,z,u)\, \Phi(x,y*z,u)\, \Phi(x,y,z), \\ a(1*x) = a(x) = a(x*1), \quad \Phi(x,1,z) = 1. \end{gathered} \tag{2.5.21}$$

Now it is clear that (2.5.21) (and respectively (2.5.18)) is the sufficient condition for the commutativity of the diagram:

$$\begin{array}{ccccc} a\left(((x*y)*z)*u\right) & \longrightarrow & a((x*y)*(z*u)) & \longrightarrow & a(x*(y*(z*u))) \\ \downarrow & & & & \downarrow \\ a((x*(y*z))*u) & & \xrightarrow{\qquad\qquad} & & a(x*((y*z)*u)) \end{array}$$

**Remark 4.** Applications of the theory of quasi-Hopf algebras to the solutions of the Knizhnik–Zamolodchikov equations are discussed in [60, 61]. On the other hand, one can suppose that, by virtue of the occurrence of the pentagonal relation (2.5.18) for the element $\Phi$, quasi-Hopf algebras will be associated with multidimensional generalizations of Yang–Baxter equations.

## 3. The Yang–Baxter equation and quantization of Lie groups

In this section, we discuss the $R$-matrix approach to the theory of quantum groups [42], on the basis of which we perform a quantization of classical Lie groups and also some Lie supergroups. We present trigonometric solutions of the Yang–Baxter equation invariant under the adjoint action of the quantum groups $GL_q(N)$, $SO_q(N)$, $Sp_q(2n)$ and supergroups $GL_q(N|M)$, $Osp_q(N|2m)$. We briefly discuss the corresponding Yangian (rational) solutions and $Z_N \otimes Z_N$ symmetric elliptic solutions of the Yang–Baxter equation. We also show that for every (trigonometric) solution $R(x)$ of the Yang–Baxter equation one can construct the set of difference equations which are called quantum Knizhnik–Zamolodchikov equations.

### *3.1. Numerical R-matrices*

This subsection is based on the results presented in [67, 68].

#### *3.1.1. Invertible and skew-invertible R-matrices*

Let $\mathcal{A}$ be a quasitriangular Hopf algebra. Consider representations $T^{(\nu)}$ of $\mathcal{A}$ in $N_\nu$-dimensional vector spaces $V_\nu$ (the index $\nu$ enumerates representations). In view of (2.2.8) and (2.3.12), the matrix $(R_{(\nu,\mu)})^{i_\nu j_\mu}_{k_\nu l_\mu} = (T^{(\nu) i_\nu}_{\;\;k_\nu} \otimes T^{(\mu) j_\mu}_{\;\;l_\mu})\mathcal{R}$, where $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ is the universal element (2.3.10), satisfies the generalized matrix Yang–Baxter equation

$$(R_{(\nu,\mu)})^{i_\nu i_\mu}_{j_\nu j_\mu}\, (R_{(\nu,\lambda)})^{j_\nu i_\lambda}_{k_\nu j_\lambda}\, (R_{(\mu,\lambda)})^{j_\mu j_\lambda}_{k_\mu k_\lambda} = (R_{(\mu,\lambda)})^{i_\mu i_\lambda}_{j_\mu j_\lambda}\, (R_{(\nu,\lambda)})^{i_\nu j_\lambda}_{j_\nu k_\lambda}\, (R_{(\nu,\mu)})^{j_\nu j_\mu}_{k_\nu k_\mu}. \tag{3.1.1}$$

Here the summation over repeated indices $j_\nu, j_\mu, j_\lambda$ is assumed. Let the representations $T^{(\nu)}$, $T^{(\mu)}$, $T^{(\lambda)}$ be equivalent to a representation $T$ which acts in $N$-dimensional vector space $V$.

In this case, according to (3.1.1), the image $R^{ij}_{k\ell} = (T^i{}_k \otimes T^j{}_\ell)\mathcal{R}$ of the universal element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ satisfies the standard matrix Yang–Baxter equation

$$R^{i_1 i_2}_{j_1 j_2}\, R^{j_1 i_3}_{k_1 j_3}\, R^{j_2 j_3}_{k_2 k_3} = R^{i_2 i_3}_{j_2 j_3}\, R^{i_1 j_3}_{j_1 k_3}\, R^{j_1 j_2}_{k_1 k_2}. \tag{3.1.2}$$

A lot of numerical solutions of the Yang–Baxter equations (3.1.1), (3.1.2) can be constructed as representations of the universal $\mathcal{R}$-matrices. However, not all numerical solutions $R$ of Eqs. (3.1.1) and (3.1.2) are images $(T^{(\nu)} \otimes T^{(\mu)})\mathcal{R}$ and $(T \otimes T)\mathcal{R}$ of the universal element $\mathcal{R}$ for some quasitriangular Hopf algebra $\mathcal{A}$. Below we consider solutions $R \in \mathrm{End}(V \otimes V)$ of the standard matrix Yang–Baxter equation (3.1.2) that are not necessarily the universal $\mathcal{R}$-matrix representations.

First, we assume that a solution $R$ of Eq. (3.1.2) is the invertible matrix

$$R^{i_1 i_2}_{k_1 \ell_2}\, (R^{-1})^{k_1 \ell_2}_{j_1 j_2} = \delta^{i_1}_{j_1}\, \delta^{i_2}_{j_2} = (R^{-1})^{i_1 i_2}_{k_1 \ell_2}\, R^{k_1 \ell_2}_{j_1 j_2}. \tag{3.1.3}$$

Note that for all images $R = (T \otimes T)\mathcal{R}$, such invertibility follows from the invertibility (2.3.14) of the universal element $\mathcal{R}$. In terms of the concise matrix notation [42], we write relations (3.1.3) and (3.1.2) in the following equivalent forms:

$$R_{12} R_{12}^{-1} = I_{12} = R_{12}^{-1} R_{12} \iff \hat{R}_{12} \hat{R}_{12}^{-1} = I_{12} = \hat{R}_{12}^{-1} \hat{R}_{12}, \tag{3.1.4}$$

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12} \implies \tag{3.1.5}$$

$$\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12} = \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23} \implies \tag{3.1.6}$$

$$\hat{R}_{23}\, \hat{R}_{12}^{-1}\, \hat{R}_{23}^{-1} = \hat{R}_{12}^{-1}\, \hat{R}_{23}^{-1}\, \hat{R}_{12}, \quad \hat{R}_{12}\, \hat{R}_{23}^{-1}\, \hat{R}_{12}^{-1} = \hat{R}_{23}^{-1}\, \hat{R}_{12}^{-1}\, \hat{R}_{23}. \tag{3.1.7}$$

Here $\hat{R} := P\,R$, the matrix $P$ is the permutation:

$$P^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}, \quad \hat{R}^{i_1 i_2}_{j_1 j_2} = (P\,R)^{i_1 i_2}_{j_1 j_2} = R^{i_2 i_1}_{j_1 j_2}, \tag{3.1.8}$$

$I_{12} := I \otimes I$ ($I \in \mathrm{Mat}(N)$ is the unit matrix in $V$) and indices $1, 2, 3$ label the vector spaces $V$ in $V^{\otimes 3}$ where the corresponding matrices $R_{12}, R_{23}, \dots$ act nontrivially, e.g., $R_{12} = R \otimes I$, $R_{23} = I \otimes R$, etc. We also note that if matrix $R_{12}$ satisfies the Yang–Baxter equation (3.1.2), then the matrix $R'_{12} = R_{21}$ also satisfies the Yang–Baxter equation

$$R_{21}\, R_{31}\, R_{32} = R_{32}\, R_{31}\, R_{21} \quad \overset{1 \leftrightarrow 3}{\iff} \quad R_{23}\, R_{13}\, R_{12} = R_{12}\, R_{13}\, R_{23}. \tag{3.1.9}$$

In what follows, we introduce matrices

$$\hat{R}_a := \hat{R}_{a,a+1} = I^{\otimes(a-1)} \otimes \hat{R} \otimes I^{\otimes(M-a)}, \quad (a = 1, \dots, M), \tag{3.1.10}$$

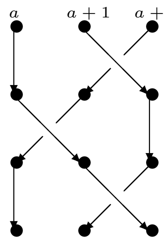
which act in the space $V^{\otimes(M+1)}$ and, according to Yang–Baxter equations (3.1.6), we have braid relations

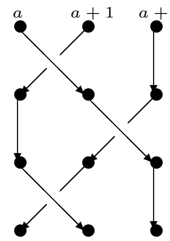
$$\hat{R}_a\, \hat{R}_{a+1}\, \hat{R}_a = \hat{R}_{a+1}\, \hat{R}_a\, \hat{R}_{a+1} \quad (a = 1, \dots, M). \tag{3.1.11}$$

In view of these relations and locality relations

$$[\hat{R}_a,\, \hat{R}_b] = 0 \ \ (\text{for} \ \ |a-b| > 1), \tag{3.1.12}$$

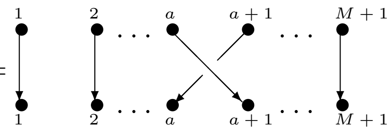
the invertible matrices $\hat{R}_a$ define a representation of generators of the braid group $\mathcal{B}_{M+1}$ (see definition in Subsection **4.1**). The name "braid group" is justified since relation (3.1.11) admits the graphic visualization

$$\hat{R}_{a+1} \cdot \hat{R}_a \cdot \hat{R}_{a+1} \; = $$



$$=$$



$$= \; \hat{R}_a \cdot \hat{R}_{a+1} \cdot \hat{R}_a \, , \qquad (3.1.13)$$

that is an identity of two braids with three strands (the third Reidemeister move). Here we make use of the graphic representation

$$\hat{R}_a \; =$$



$$. \qquad (3.1.14)$$

We discuss in detail the braid group $\mathcal{B}_{M+1}$, its group algebra $\mathbb{C}[\mathcal{B}_{M+1}]$ and finite dimensional quotients of $\mathbb{C}[\mathcal{B}_{M+1}]$ in Section 4 below.

Let $X(\hat{R}_a)$ be a formal series in $\hat{R}_a^{\pm 1}$. The direct consequences of (3.1.11) are equations

$$X(\hat{R}_a)\, \hat{R}_{a+1}\, \hat{R}_a = \hat{R}_{a+1}\, \hat{R}_a\, X(\hat{R}_{a+1}), \;\; \hat{R}_a\, \hat{R}_{a+1}\, X(\hat{R}_a) = X(\hat{R}_{a+1})\, \hat{R}_a\, \hat{R}_{a+1}, \qquad (3.1.15)$$

which make it possible to carry functions $X(\hat{R}_a)$, $X(\hat{R}_{a+1})$ through the operators $\hat{R}_{a+1}\hat{R}_a$ and $\hat{R}_a\hat{R}_{a+1}$.

**Definition 8.** *The matrix $R \in \mathrm{End}(V^{\otimes 2})$ is called skew-invertible if there exists matrix $\Psi \in \mathrm{End}(V^{\otimes 2})$ such that (cf. (3.1.3))*

$$R^{mk}_{\;\;jn}\, \Psi^{in}_{\;\;ml} = \delta^i_j \delta^k_l = \Psi^{ni}_{\;\;lm}\, R^{km}_{\;\;nj}. \qquad (3.1.16)$$

*The index-free forms of these relations are*[4]

$$R^{t_1}_{12}\, \Psi^{t_1}_{12} = I_{12}, \quad \Psi^{t_1}_{12}\, R^{t_1}_{12} = I_{12}, \qquad (3.1.17)$$

$$\mathrm{Tr}_2(\hat{R}_{12}\, \hat{\Psi}_{23}) = P_{13} = \mathrm{Tr}_2\left(\hat{\Psi}_{12}\, \hat{R}_{23}\right), \qquad (3.1.18)$$

*where $\hat{\Psi} = P\,\Psi$. We say that the invertible and skew-invertible $R$-matrix is completely invertible if the inverse matrix $R^{-1}$ is also skew-invertible, i.e., there exists a matrix $\Phi \in \mathrm{End}(V^{\otimes 2})$ such that*

$$\Phi^{t_2}_{12}\, (R^{-1})^{t_2}_{12} = I_{12} = (R^{-1})^{t_2}_{12}\, \Phi^{t_2}_{12} \;\; \Leftrightarrow \;\; \Phi^{i_1 i_2}_{\;\;k_2 j_1}\, (R^{-1})^{k_2 i_3}_{\;\;j_3 i_2} = \delta^{i_1}_{j_3}\, \delta^{i_3}_{j_1} = (R^{-1})^{i_1 i_2}_{\;\;k_2 j_1}\, \Phi^{k_2 i_3}_{\;\;j_3 i_2} \;\; \Rightarrow$$

$$\mathrm{Tr}_2\left(\hat{\Phi}_{12}\, \hat{R}^{-1}_{23}\right) = P_{13} = \mathrm{Tr}_2(\hat{R}^{-1}_{12}\, \hat{\Phi}_{23}), \qquad (3.1.19)$$

*where $\hat{R}^{-1} = R^{-1}\,P$ and $\hat{\Phi} = \Phi\,P$.*

The skew-invertible $R$-matrices were considered in [67], where operator $\Psi_{12}$ was denoted as $((R^{t_1}_{12})^{-1})^{t_1}$ (cf. (3.1.17)).

[4] The form (3.1.18) is very convenient for calculations (see below) and was proposed in [97]. Equations (3.1.17) are equivalently written as $\Psi^{t_2}_{12}\, R^{t_2}_{12} = I_{12} = R^{t_2}_{12}\, \Psi^{t_2}_{12}$.

*3.1.2. Quantum traces*

Now we define four matrices

$$D_1 = \mathrm{Tr}_2(\hat{\Psi}_{12}), \quad Q_2 = \mathrm{Tr}_1(\hat{\Psi}_{12}), \tag{3.1.20}$$

$$\overline{D}_1 = \mathrm{Tr}_2(\hat{\Phi}_{12}), \quad \overline{Q}_2 = \mathrm{Tr}_1(\hat{\Phi}_{12}), \tag{3.1.21}$$

which are important for our consideration below.

**Proposition 3.3.** *Let the Yang–Baxter matrix $R$ be invertible and skew-invertible, then the following identities hold:*

$$\mathrm{Tr}_2(\hat{R}_{12} D_2) = I_1, \quad \mathrm{Tr}_1(Q_1\, \hat{R}_{12}) = I_2, \tag{3.1.22}$$

$$\mathrm{Tr}_2(\hat{R}^{-1}_{12}\, \overline{D}_2) = I_1, \quad \mathrm{Tr}_1(\overline{Q}_1\, \hat{R}^{-1}_{12}) = I_2, \tag{3.1.23}$$

$$D_0\, P_{02} = \mathrm{Tr}_3 D_3 \hat{R}^{-1}_{23}\, \hat{R}_{03}, \quad D_0\, P_{02} = \mathrm{Tr}_3 D_3 \hat{R}_{23}\, \hat{R}^{-1}_{03}, \tag{3.1.24}$$

$$Q_0\, P_{02} = \mathrm{Tr}_1 Q_1 \hat{R}^{-1}_{12}\, \hat{R}_{10}, \quad Q_0\, P_{02} = \mathrm{Tr}_1 Q_1 \hat{R}_{12}\, \hat{R}^{-1}_{10}, \tag{3.1.25}$$

$$D_2\, \hat{R}^{-1}_{12} = \hat{\Psi}_{21}\, D_1, \quad \hat{R}^{-1}_{12}\, D_2 = D_1\, \hat{\Psi}_{21}, \tag{3.1.26}$$

$$Q_1\, \hat{R}^{-1}_{12} = \hat{\Psi}_{21}\, Q_2, \quad \hat{R}^{-1}_{12}\, Q_1 = Q_2\, \hat{\Psi}_{21}, \tag{3.1.27}$$

*where the matrices $D$ and $Q$ commute and satisfy*

$$D_2\, Q_2 = Q_2\, D_2 = \mathrm{Tr}_3(D_3\, \hat{R}^{-1}_2) = \mathrm{Tr}_1(Q_1\, \hat{R}^{-1}_1). \tag{3.1.28}$$

*If the matrix $R$ is completely invertible, then*

$$\hat{R}_{12}\, D_2 = D_1\, \hat{\Phi}_{21}, \quad D_2\, \hat{R}_{12} = \hat{\Phi}_{21}\, D_1. \tag{3.1.29}$$

$$\hat{R}_{12}\, Q_1 = Q_2\, \hat{\Phi}_{21}, \quad Q_1\, \hat{R}_{12} = \hat{\Phi}_{21}\, Q_2, \tag{3.1.30}$$

*and the matrices $D$ and $Q$ are invertible such that*

$$D^{-1} = \overline{Q}, \quad Q^{-1} = \overline{D}. \tag{3.1.31}$$

*Conversely, if the matrix $D$ (or $Q$) is invertible, then the matrix $R$ is completely invertible. For invertible matrices $D$ and $Q$ one has the relations*

$$\mathrm{Tr}_1(D^{-1}_1\, \hat{R}^{-1}_{12}) = I_2 = \mathrm{Tr}_3(Q^{-1}_3\, \hat{R}^{-1}_{23}). \tag{3.1.32}$$

**Proof.** Identities (3.1.22) and (3.1.23) follow from (3.1.18) and (3.1.19). To obtain (3.1.24) and (3.1.25), we multiply both sides of Eqs. (3.1.15) (for $a = 1$) from the left by $\hat{\Psi}_{01}$ and from the right by $\hat{\Psi}_{34}$ and take the trace $\mathrm{Tr}_{13}$ ($\equiv \mathrm{Tr}_1\, \mathrm{Tr}_3$). Using (3.1.18), we obtain

$$\mathrm{Tr}_1 \hat{\Psi}_{01} X(\hat{R}_1) P_{24} \hat{R}_1 = \mathrm{Tr}_3 \hat{R}_2\, P_{02}\, X(\hat{R}_2)\, \hat{\Psi}_{34}, \tag{3.1.33}$$

$$\mathrm{Tr}_1 \hat{\Psi}_{01} \hat{R}_1\, P_{24}\, X(\hat{R}_1) = \mathrm{Tr}_3 X(\hat{R}_2)\, P_{02}\, \hat{R}_2\, \hat{\Psi}_{34}, \tag{3.1.34}$$

where $\hat{R}_a \equiv \hat{R}_{a\,a+1}$ (see (3.1.10)). We put $X(\hat{R}) = \hat{R}^{-1}$ in (3.1.33), (3.1.34) and take the traces $\mathrm{Tr}_4$ or $\mathrm{Tr}_0$. Using (3.1.18) and (3.1.20), we obtain four identities (dependent on different choices of $\pm$)

$$D_0\, I_2 = \mathrm{Tr}_3 D_3 \hat{R}^{\mp 1}_2\, P_{02}\, \hat{R}^{\pm 1}_2, \quad Q_0\, I_2 = \mathrm{Tr}_1 Q_1 \hat{R}^{\pm 1}_1\, P_{02}\, \hat{R}^{\mp 1}_1, \tag{3.1.35}$$

which can be easily written as (3.1.24) and (3.1.25). Applying to both sides of the first relation in (3.1.24) the operation $\mathrm{Tr}_0(\hat{\Psi}_{10}\dots)$ and to both sides of the second relation in (3.1.24) the operation $\mathrm{Tr}_2(\hat{\Psi}_{12}\dots)$, we obtain identities

$$\mathrm{Tr}_0(\hat{\Psi}_{10}D_0P_{02}) = \mathrm{Tr}_{03}(\hat{\Psi}_{10}D_3\hat{R}_{23}^{-1}\hat{R}_{03}), \quad \mathrm{Tr}_2(D_0\hat{\Psi}_{12}P_{02}) = \mathrm{Tr}_{23}(\hat{\Psi}_{12}D_3\hat{R}_{23}\hat{R}_{03}^{-1}),$$

which, by means of (3.1.18), give (3.1.26). Similarly, applying to both sides of the first relation in (3.1.25) the operation $\mathrm{Tr}_0(\dots\hat{\Psi}_{03})$ and to both sides of the second relation in (3.1.25) the operation $\mathrm{Tr}_2(\dots\hat{\Psi}_{23})$, we obtain (3.1.27). Taking the traces $\mathrm{Tr}_2(\dots)$ and $\mathrm{Tr}_1(\dots)$ of (3.1.26) and (3.1.27), respectively, we deduce (3.1.28).

If the matrix $R$ is completely invertible, then acting to the first relation (3.1.24) by $\mathrm{Tr}_2(\hat{\Phi}_{12}\dots)$ and to the second relation (3.1.24) by $\mathrm{Tr}_0(\hat{\Phi}_{10}\dots)$, we obtain (3.1.29). Analogously, acting to the first relation (3.1.25) by $\mathrm{Tr}_2(\dots\hat{\Phi}_{23})$ and to the second relation (3.1.25) by $\mathrm{Tr}_0(\dots\hat{\Phi}_{03})$, we find (3.1.30). Equations (3.1.31) are obtained by taking traces $\mathrm{Tr}_2(\dots)$ and $\mathrm{Tr}_1(\dots)$ of (3.1.29) and (3.1.30), respectively, and applying (3.1.21), (3.1.22). Thus, for the completely invertible $R$ the matrices $D$ and $Q$ are invertible.

Conversely, if the matrix $D$ is invertible, then $D_1\hat{R}_{21}D_2^{-1}$ (cf. (3.1.29)) is the skew-inverse matrix for $\hat{R}^{-1}$. Indeed,

$$\mathrm{Tr}_2\left(\hat{R}_{12}^{-1}\,D_2\,\hat{R}_{32}\,D_3^{-1}\right) = \mathrm{Tr}_2\left(\hat{R}_{12}^{-1}\,D_2\,\hat{R}_{32}\right)\,D_3^{-1} =$$

$$= D_1\,\mathrm{Tr}_2\left(\hat{\Psi}_{21}\,\hat{R}_{32}\right)\,D_3^{-1} = D_1\,P_{13}\,D_3^{-1} = P_{13},$$

where in the second equality we apply the second relation in (3.1.26). Thus, the $R$-matrix is completely invertible. For the invertible matrix $Q$ the proof of the fact that the Yang–Baxter $R$-matrix is completely invertible is similar. For invertible matrices $D$ and $Q$ we have (3.1.31) and one can rewrite relations (3.1.23) as (3.1.32). ■

**Corollary 1.** Let $\hat{R}$ be skew-invertible and the matrix $A_{12}$ be one of the matrices $\{\hat{R}_{12},\,\hat{R}_{12}^{-1},\,\hat{\Psi}_{12}\}$. Then from (3.1.26) and (3.1.27) we obtain

$$[A_{12},\,D_1\,D_2] = 0 = [A_{12},\,Q_1\,Q_2], \tag{3.1.36}$$

$$A_{12}\,(D\,Q)_1 = (D\,Q)_2\,A_{12}. \tag{3.1.37}$$

If $\hat{R}$ is completely invertible, then matrices $\hat{\Psi}_{12}$, $\hat{\Phi}_{12}$ are invertible

$$\hat{\Psi}_{12}^{-1} = D_1^{-1}\hat{R}_{21}D_2 = Q_2^{-1}\hat{R}_{21}Q_1, \quad \hat{\Phi}_{12}^{-1} = D_2\hat{R}_{21}^{-1}D_1^{-1} = Q_1\hat{R}_{21}^{-1}Q_2^{-1}.$$

In this case, by using (3.1.29) and (3.1.30), we prove Eqs. (3.1.36), (3.1.37) for $A_{12} = \hat{\Phi}_{12}$ and deduce the relation on the matrices $\hat{\Phi}$ and $\hat{\Psi}$:

$$\hat{\Phi}_{12}^{-1} = D_2^2\,\hat{\Psi}_{12}\,D_1^{-2} = Q_1^2\,\hat{\Psi}_{12}\,Q_2^{-2}.$$

**Corollary 2.** For any quantum $(N\times N)$ matrix $E$ (with noncommutative entries $E^i_j$) one can find the following identities:

$$\mathrm{Tr}(D\,E)\,I_1 = \mathrm{Tr}_2\left(D_2\hat{R}_1^{\mp1}\,E_1\,\hat{R}_1^{\pm1}\right), \quad \mathrm{Tr}(Q\,E)\,I_2 = \mathrm{Tr}_1\left(Q_1\hat{R}_1^{\pm1}\,E_2\,\hat{R}_1^{\mp1}\right) \tag{3.1.38}$$

that demonstrate the invariance properties of the quantum traces

$$\mathrm{Tr}(D\,E) =: \mathrm{Tr}_{\mathcal{D}}(E), \quad \mathrm{Tr}(Q\,E) =: \mathrm{Tr}_{\mathcal{Q}}(E). \tag{3.1.39}$$

To prove identities (3.1.38), we multiply Eqs. (3.1.35) by the matrix $E_0$ and take the trace $\mathrm{Tr}_0(\dots)$. Note that in view of (3.1.36), the multiple quantum traces satisfy cyclic property

$$\begin{aligned} \mathrm{Tr}_{\mathcal{D}(1\dots m)}\left(X(\hat{R})\cdot Y\right) &= \mathrm{Tr}_{\mathcal{D}(1\dots m)}\left(Y\cdot X(\hat{R})\right), \\ \mathrm{Tr}_{\mathcal{Q}(1\dots m)}\left(X(\hat{R})\cdot Y\right) &= \mathrm{Tr}_{\mathcal{Q}(1\dots m)}\left(Y\cdot X(\hat{R})\right), \end{aligned} \tag{3.1.40}$$

where $X(\hat{R}) \in \mathrm{End}(V^{\otimes m})$ denotes arbitrary element of the group algebra of the braid group $\mathcal{B}_m$ in $R$-matrix representation (3.1.11), (3.1.12) and $Y \in \mathrm{End}(V^{\otimes m})$ are arbitrary (quantum) operators.

**Corollary 3.** Let $R$ be completely invertible matrix. We multiply the first and the second Yang–Baxter equations in (3.1.7), respectively, from the right and the left by the matrix $D_3$:

$$\hat{R}_2\,\hat{R}_1^{-1}\,\hat{R}_2^{-1}\,D_3 = \hat{R}_1^{-1}\,\hat{R}_2^{-1}\,\hat{R}_1\,D_3, \quad D_3\,\hat{R}_1\,\hat{R}_2^{-1}\,\hat{R}_1^{-1} = D_3\,\hat{R}_2^{-1}\,\hat{R}_1^{-1}\,\hat{R}_2,$$

and use relations (3.1.26). As a result, we deduce

$$\hat{R}_{23}\,\hat{\Psi}_{21}\,\hat{\Psi}_{32} = \hat{\Psi}_{21}\,\hat{\Psi}_{32}\,\hat{R}_{12}, \quad \hat{R}_{12}\,\hat{\Psi}_{32}\,\hat{\Psi}_{21} = \hat{\Psi}_{32}\,\hat{\Psi}_{21}\,\hat{R}_{23}. \tag{3.1.41}$$

Analogously, if we multiply Yang–Baxter equations (3.1.15) (for $X(\hat{R}_a) = \hat{R}_a^{-1}$ and $a = 1$) from the left and the right by the matrix $Q_1$ and use relations (3.1.30), we respectively deduce

$$\hat{\Phi}_{21}\,\hat{\Phi}_{32}\,\hat{R}_{12} = \hat{R}_{23}\,\hat{\Phi}_{21}\,\hat{\Phi}_{32}, \quad \hat{\Phi}_{32}\,\hat{\Phi}_{21}\,\hat{R}_{23} = \hat{R}_{12}\,\hat{\Phi}_{32}\,\hat{\Phi}_{21}. \tag{3.1.42}$$

**Corollary 4.** The trace $\mathrm{Tr}_{04}(\dots)$ of Eq. (3.1.33) (or (3.1.34)) gives

$$\mathrm{Tr}_1 Q_1 X(\hat{R}_1) = \mathrm{Tr}_3 D_3 X(\hat{R}_2) \equiv Y_2(X), \tag{3.1.43}$$

where we redefined the arbitrary function $X$: $X(\hat{R})\hat{R} \to X(\hat{R})$. In particular, for $X = 1$ we obtain $\mathrm{Tr}(D) = \mathrm{Tr}(Q)$. Equation (3.1.43) leads to the following identity:

$$\mathrm{Tr}_{12}\left(Q_1 Q_2 X(\hat{R}_1)\right) = \mathrm{Tr}_{23}\left(D_3 Q_2 X(\hat{R}_2)\right) = \mathrm{Tr}_{34}\left(D_3 D_4 X(\hat{R}_3)\right). \tag{3.1.44}$$

**Proposition 3.4.** *For any polynomial $X \in \mathbb{C}[\hat{R}_1, \hat{R}_1^{-1}]$ the matrix $Y(X)$ defined in (3.1.43) satisfies $[D, Y] = 0 = [Y, Q]$ and*

$$Y_2(X)\,\hat{R}_1^{\pm 1} = \hat{R}_1^{\pm 1}\,Y_1(X). \tag{3.1.45}$$

*Matrices*

$$Y_2^{(n)} := Y(\hat{R}^n) = \mathrm{Tr}_3\left(D_3\,\hat{R}_2^n\right) = \mathrm{Tr}_1\left(Q_1\,\hat{R}_1^n\right), \qquad \forall\, n \in \mathbb{Z}, \tag{3.1.46}$$

*generate a commutative set.*

**Proof.** From (3.1.43) and (3.1.36) we have

$$D_2\, Y_2 = \mathrm{Tr}_3\big(D_2\, D_3\, X(\hat{R}_2)\big) = \mathrm{Tr}_3\big(X(\hat{R}_2)\, D_2\, D_3\big) = Y_2\, D_2,$$
$$Q_2\, Y_2 = \mathrm{Tr}_1\big(Q_1\, Q_2\, X(\hat{R}_1)\big) = \mathrm{Tr}_3\big(X(\hat{R}_1)\, Q_1\, Q_2\big) = Y_2\, Q_2.$$

The left-hand side of (3.1.45) is transformed as follows:

$$Y_2(X)\ \hat{R}_1^{\pm 1} = \mathrm{Tr}_3\big(D_3 X(\hat{R}_2)\hat{R}_1^{\pm 1}\hat{R}_2^{\pm 1}\hat{R}_2^{\mp 1}\big) = \hat{R}_1^{\pm 1}\ \mathrm{Tr}_3\big(D_3 \hat{R}_2^{\pm 1} X(\hat{R}_1)\hat{R}_2^{\mp 1}\big) =$$
$$= \hat{R}_1^{\pm 1}\ \mathrm{Tr}_2\big(D_2\, X(\hat{R}_1)\big) = \hat{R}_1^{\pm 1}\ Y_1(X),$$

where we used (3.1.15) and the first relation in (3.1.38).

The commutativity of the matrices $Y_2^{(n)}$ follows from (3.1.45), since for even and odd $n$ we have, respectively

$$Y_2(X)\, Y_2^{(2k)} = \mathrm{Tr}_3\left(D_3\, Y_2\, \hat{R}_2^{2k}\right) = \mathrm{Tr}_3\left(D_3\, \hat{R}_2^{2k}\, Y_2\right) = Y_2^{(2k)}\, Y_2(X),$$

$$Y_2(X)\, Y_2^{(2k+1)} = \mathrm{Tr}_1\left(Q_1\, Y_2\, \hat{R}_1^{2k+1}\right) = \mathrm{Tr}_1\left(Q_1\, \hat{R}_1^{2k+1}\, Y_1\right) = \mathrm{Tr}_1\left(Y_1\, Q_1\, \hat{R}_1^{2k+1}\right) =$$
$$= \mathrm{Tr}_1\left(Q_1\, Y_1\, \hat{R}_1^{2k+1}\right) = \mathrm{Tr}_1\left(Q_1\, \hat{R}_1^{2k+1}\, Y_2\right) = Y_2^{(2k+1)}\, Y_2(X).$$

For $X(\hat{R}) = \hat{R}^m$ $(m \in \mathbb{Z})$ we obtain commutativity of matrices (3.1.46). ∎

**Proposition 3.5.** *The identity (3.1.44) is generalized as*

$$\mathrm{Tr}_{1\dots n}(Q_1 \cdots Q_k D_{k+1} \cdots D_n X_{1\to n}) = \mathrm{Tr}_{1\dots n}(D_1 \cdots D_n X_{1\to n}) \quad (\forall n \geqslant 2,\ k = 1, \dots, n), \tag{3.1.47}$$

*where* $X_{1\to n} := X(\hat{R}_1, \dots, \hat{R}_{n-1}) \in \mathbb{C}[\hat{R}_1^{\pm 1}, \dots, \hat{R}_{n-1}^{\pm 1}]$ *is an element of the group algebra of the braid group* $\mathcal{B}_n$ *in the R-matrix representation*[5] *(3.1.11), (3.1.12).*

**Proof.** Indeed, from (3.1.11), (3.1.12) we have $\hat{R}_1 \cdots \hat{R}_n\, X_{1\to n} = X_{2\to n+1}\, \hat{R}_1 \cdots \hat{R}_n$. Multiplying both sides of this equation by the matrices $Q_1$ and $D_{n+1}$ from the left and the right and taking the trace $\mathrm{Tr}_1 \mathrm{Tr}_{n+1}(\dots)$, we deduce (by means of (3.1.22))

$$\mathrm{Tr}_1\left(Q_1\, \hat{R}_1 \cdots \hat{R}_{n-1}\, X_{1\to n}\right) = \mathrm{Tr}_{n+1}\left(D_{n+1}\, X_{2\to n+1}\, \hat{R}_2 \cdots \hat{R}_n\right),$$

which is written, after the redefinition $X_{1\to n} \to (\hat{R}_1 \cdots \hat{R}_{n-1})^{-1}\, X_{1\to n}$, in the form

$$\mathrm{Tr}_1\left(Q_1\, X_{1\to n}\right) = \mathrm{Tr}_{n+1}\left(D_{n+1}\, (\hat{R}_2 \cdots \hat{R}_n)^{-1}\, X_{2\to n+1}\, \hat{R}_2 \cdots \hat{R}_n\right). \tag{3.1.48}$$

Then, applying the trace $\mathrm{Tr}_2(Q_2 \dots)$ to (3.1.48) (and again using (3.1.48)), we obtain

$$\begin{aligned} \mathrm{Tr}_{12}\left(Q_1 Q_2\, X_{1\to n}\right) &= \mathrm{Tr}_{n+1} D_{n+1} \mathrm{Tr}_2\left(Q_2\, (\hat{R}_2 \cdots \hat{R}_n)^{-1}\, X_{2\to n+1}\, \hat{R}_2 \cdots \hat{R}_n\right) = \\ &= \mathrm{Tr}_{n+1,n+2}\left(D_{n+1} D_{n+2}\, (\hat{R}_3 \cdots \hat{R}_{n+1})^{-2}\, X_{3\to n+2}\, (\hat{R}_3 \cdots \hat{R}_{n+1})^2\right). \end{aligned} \tag{3.1.49}$$

[5] In view of the graphical representation (3.1.14), any monomial in $X_{1\to n}$ is interpreted as a braid with $n$ strands.

Applying the trace $\mathrm{Tr}_3(Q_3\dots)$ to (3.1.49), etc., we obtain

$$\begin{gathered}\mathrm{Tr}_{1\dots k}\left(Q_1\cdots Q_k\, X_{1\to n}\right)= \\ =\mathrm{Tr}_{n+1\dots n+k}\Big(D_{n+1}\cdots D_{n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^{-k}\,X_{k+1\to n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^{k}\Big)\,,\end{gathered}\tag{3.1.50}$$

and finally multiplying both sides of (3.1.50) by $D_{k+1}\cdots D_n$ from the left and taking the trace $\mathrm{Tr}_{k+1\dots n}$ (applying $\mathrm{Tr}_{k+1\dots n}(D_{k+1}\cdots D_n\,\dots)$ to both sides of (3.1.50)), we deduce (3.1.47)

$$\begin{gathered}\mathrm{Tr}_{1\dots n}\left(Q_1\cdots Q_k D_{k+1}\cdots D_n\, X_{1\to n}\right)= \\ =\mathrm{Tr}_{k+1\dots n+k}\Big(D_{k+1}\cdots D_{n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^{-k}\,X_{k+1\to n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^{k}\Big)= \\ =\mathrm{Tr}_{k+1\dots n+k}\left(D_{k+1}\cdots D_{n+k}\, X_{k+1\to n+k}\right),\end{gathered}\tag{3.1.51}$$

where we have used the cyclic property (3.1.40). ∎

**Remark 1.** A numerical $R$-matrix which is the image $(T\otimes T)\mathcal{R}$ of the universal $\mathcal{R}$-matrix (2.5.9) for the quasitriangular Hopf algebra is obliged to be skew-invertible. Indeed, relations (2.5.10) are written in the matrix form

$$\begin{aligned}\delta^i_j\delta^k_\ell &= T^i_j(\alpha_\mu\alpha_\nu)\,T^k_\ell(\beta_\nu S(\beta_\mu)) = R^{mk}_{j\,n}\;\;T^i_m(\alpha_\mu)\,T^n_\ell(S(\beta_\mu)),\\ \delta^k_\ell\delta^i_j &= T^k_\ell(\alpha_\mu\alpha_\nu)\,T^i_j(S(\beta_\nu)\beta_\mu) = T^n_\ell(\alpha_\nu)\,T^i_m(S(\beta_\nu))\;\;R^{km}_{nj},\end{aligned}$$

and, thus, relations (2.5.10) are the algebraic counterparts of (3.1.16), where the matrix $\Psi$ is given by the equation

$$\Psi^{in}_{ml} = T^i_m(\alpha_\mu)\,T^n_l(S(\beta_\mu)) = \hat{\Psi}^{ni}_{ml}.\tag{3.1.52}$$

Moreover, in view of (2.3.6), the transposed matrix $\Psi^{t_2}$ of (3.1.52) is interpreted as the image $(T\otimes\overline{T})\mathcal{R}$, where $\overline{T}$ denotes a contragredient representation to $T$, i.e., $\overline{T}(a)=T^t(S(a))$ $(\forall a\in\mathcal{A})$. Then the second equation in (3.1.41) is nothing but the image of the universal Yang–Baxter equation (2.3.12) in the representation $(T\otimes T\otimes\overline{T})$.

The image $(T\otimes T)\mathcal{R}^{-1}=R^{-1}$ is also skew-invertible. The matrix $\Phi_{12}$ in (3.1.19) is given by

$$\Phi^{in}_{ml} = T^i_m(S(\gamma_\mu))\,T^n_l(\delta_\mu) = \hat{\Phi}^{in}_{lm}\tag{3.1.53}$$

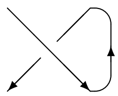
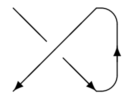
and the algebraic counterpart of (3.1.19) is (2.5.11). The second equation in (3.1.42) is the image of the universal equation $\mathcal{R}_{23}\mathcal{R}^{-1}_{12}\mathcal{R}^{-1}_{13}=\mathcal{R}^{-1}_{13}\mathcal{R}^{-1}_{12}\mathcal{R}_{23}$ (see (2.3.12)) in the representation $(\overline{T}\otimes T\otimes T)$. From Eqs. (3.1.52), (3.1.53) we also have the universal formulas for matrices (3.1.20), (3.1.21) of quantum traces

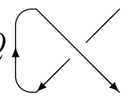
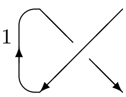
$$\begin{gathered}D=T(S(\beta_\mu)\alpha_\mu)=T(u),\quad \overline{D}=T(S(\gamma_\mu)\delta_\mu)=T(u_2),\\ Q=T(\alpha_\mu S(\beta_\mu))=T(S(u_3))=T(u_4^{-1}),\quad \overline{Q}=T(\delta_\mu S(\gamma_\mu))=T(S(u_4))=T(u_3^{-1}),\end{gathered}\tag{3.1.54}$$

where elements $u_1=u,u_2,u_3,u_4$ were introduced in (2.5.15) in Subsection **2.5**. Then, in view of Proposition **2.2**, all matrices (3.1.54) commute with each other and the products $DQ$, $D\overline{Q}=I$, $\overline{D}Q=I$, and $\overline{D}\,\overline{Q}=(DQ)^{-1}$ are images of central elements $(u_iu_j^{-1})\in\mathcal{A}$ in the representation $T$.

### *3.1.3. R-matrix formulation of link and knot invariants*

The $R$-matrix formulation of link and knot invariants was developed in [58, 59, 67] (see also references therein). Taking into account the fact that $R$-matrices satisfy (by definition) the third Reidemeister move (3.1.13), we see that Propositions **3.3** and **3.5** are important for constructing of link and knot invariants. Indeed, using graphic representation (3.1.14), one can visualize relations (3.1.22) and (3.1.32) from Proposition **3.3** as the first Reidemeister moves:

$\mathrm{Tr}_2\left(\hat{R}_{12}D_2\right) =$  $D \;=\; \downarrow I_1$ , $\qquad \mathrm{Tr}_2\left(\hat{R}^{-1}_{12}Q^{-1}_2\right) =$  $Q^{-1} \;=\; \downarrow I_1$ , (3.1.55)

$\mathrm{Tr}_1\left(Q_1\hat{R}_{12}\right) = Q$  $= \;\downarrow I_2$ , $\qquad \mathrm{Tr}_2\left(D^{-1}_1\hat{R}^{-1}_{12}\right) = D^{-1}$  $= \;\downarrow I_2$ . (3.1.56)

These pictures show that the elementary braids $\hat{R}$ and $\hat{R}^{-1}$ are closed by matrices $D$, $Q^{-1} = \overline{D}$ on the right, and by matrices $D^{-1}$, $Q = \overline{D}^{-1}$ on the left, to obtain trivial braids. We note that in general $D \neq Q^{-1}$. We stress, however, that for many explicit numerical $R$-matrices we have[6] $Q^{-1} \sim D$, and therefore, after the special normalization of $R$-matrices, we deal with the standard first Reidemeister moves. Finally, for the case of the skew-invertible $R$-matrices, Proposition **3.5** demonstrates the equivalence of the complete closuring of braids[7] $X_{1\to n}$ from the left and from the right by means of the quantum traces respectively with matrices $Q$ and $D$. Thus, for any braid $X_{1\to n}$ with $n$ strands ($X_{1\to n}$ is a monomial constructed as a product of any number of $R$-matrices $\{\hat{R}_1, \dots, \hat{R}_{n-1}\}$) the characteristic (3.1.47)

$$\mathsf{Q}(X_{1\to n}) := \mathrm{Tr}_{1\dots n}(Q_1 \cdots Q_n X_{1\to n}) \equiv \mathrm{Tr}_{1\dots n}(D_1 \cdots D_n X_{1\to n}), \tag{3.1.57}$$

gives (by closing of the braid $X_{1\to n}$) the invariant for link/knot.

**Remark 2.** Let $T$ be the representation of the quasitriangular Hopf algebra $\mathcal{A}$ in the space $V$. Consider a special matrix representation of the universal $\mathcal{R}$-matrix

$$R_{(k,m)} = \sum_{\mu} T^{\otimes k}(\alpha_\mu) \otimes T^{\otimes m}(\beta_\mu) \equiv (T^{\otimes k} \otimes T^{\otimes m})\,\mathcal{R}, \tag{3.1.58}$$

where $T^{\otimes k}$ acts to the first factor in $\mathcal{R}$, $T^{\otimes m}$ acts to the second factor in $\mathcal{R}$ and we have used the notation (2.5.9). Then applying (2.3.9), we deduce

$$R_{(k,m)} = R_{1\to k;k+m} \cdots R_{1\to k;k+2} \cdot R_{1\to k;k+1} = (P_{1\to k;k+m} \cdots P_{1\to k;k+1})\,\hat{R}_{(k,m)}, \tag{3.1.59}$$

$$\hat{R}_{(k,m)} = \hat{R}_{(m\to k+m-1)} \cdots \hat{R}_{(2\to k+1)}\,\hat{R}_{(1\to k)}, \tag{3.1.60}$$

where

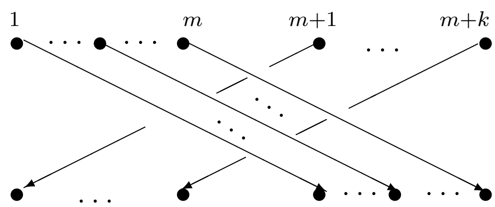
$$R_{1\to k;k+\ell} := R_{1,k+\ell}\,R_{2,k+\ell} \cdots R_{k,k+\ell} = P_{1\to k;k+\ell} \cdot \hat{R}_{(1\to k-1)}\hat{R}_{k,k+\ell},$$

$$P_{1\to k;k+\ell} := P_{1,k+\ell}\,P_{2,k+\ell} \cdots P_{k,k+\ell}, \quad \hat{R}_{(k\to\ell)} := \hat{R}_k\,\hat{R}_{k+1} \cdots \hat{R}_\ell,$$

[6] For $R = (T \otimes T)\mathcal{R}$ the matrix $DQ = T(u_1 u_4^{-1})$ is the image of central element and for irreducible representation $T$ we have $DQ \sim I$; see also Examples 1 and 2 below.

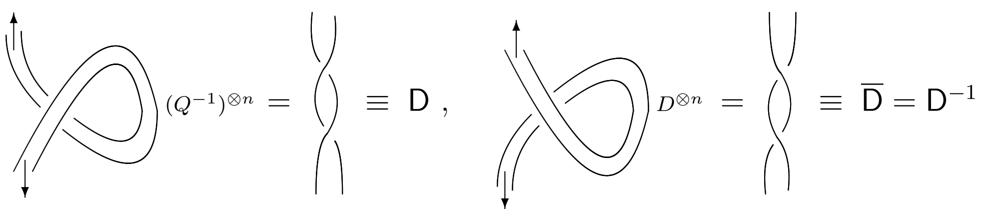
[7] Here the braids $X_{1\to n}$ are elements of the braid group $\mathcal{B}_n$ in the $R$-matrix representations.

$R_{ij} := ((T \otimes T)\mathcal{R})_{ij}$ and the braid $\hat{R}_{(k,m)}$ is obtained from matrix $R_{(k,m)}$ by substitution $R_{ij} = P_{ij}\,\hat{R}_{ij}$ and shifting all permutation matrices $P_{ij}$ to the left. The braid $\hat{R}_{(k,m)}$ defined in (3.1.60) can be visualized, by means of (3.1.14), as the intersection of two cables (or two ribbons) with $m$ and $k$ strands:

$\hat{R}_{(k,m)} \;=$ 

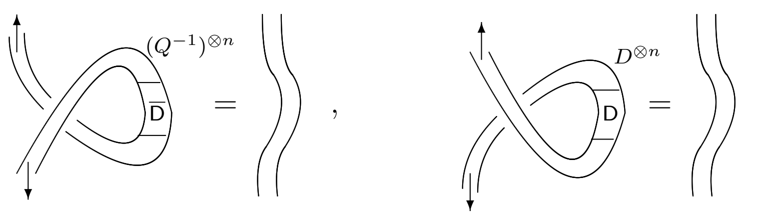
(3.1.61)

This pictorial presentation demonstrates the fact that in general the $R$-matrix approach could describe invariants not only for ordinary links and knots, but also for ribbon (cable) links and knots. In this case, the right (or left) closuring for braids with matrices $D_1 \cdots D_n \sim Q_1^{-1} \cdots Q_n^{-1}$ (or $Q^{\otimes n} \sim (D^{-1})^{\otimes n}$) is also different for the cable (ribbon) braids $\hat{R}_{(n,n)}$ and $\hat{R}^{-1}_{(n,n)}$ (cf. (3.1.55), (3.1.56)). For example, for the right closuring it follows from the visualization of the moves which are shown in the pictures:



(3.1.62)

where we pull the ribbons along the arrows on the left-hand side (l. h. s.) of the equalities and obtain two differently twisted ribbons (as spirals) in the right-hand side (r. h. s.) of the equalities. Thus, for ribbon (cable) links/knots, to obtain the first Reidemeister moves, we need to insert matrices $\mathsf{D}$ and $\overline{\mathsf{D}}$ in the closuring of braids $D^{\otimes n} \cdot \mathsf{D}$ and $(Q^{-1})^{\otimes n} \cdot \overline{\mathsf{D}}$ (here the "ribbon" matrices $\mathsf{D}$ and $\overline{\mathsf{D}}$ are defined in (3.1.62)):



(3.1.63)

Thus, in the right-hand side of the relations (3.1.63), we obtain the unit operators in $V^{\otimes n}$.

*3.1.4. Spectral decomposition of R-matrices and examples of knot/link invariants*

We now assume that the invertible Yang–Baxter $\hat{R}$-matrix obeys the characteristic equation

$$(\hat{R} - \mu_1)(\hat{R} - \mu_2) \cdots (\hat{R} - \mu_M) = 0, \tag{3.1.64}$$

where $\mu_i \in \mathbb{C}$, $\mu_i \neq \mu_j$ if $i \neq j$ and $\mu_i \neq 0\ \forall i$. This equation can be represented in the form

$$\hat{R}^M - \sigma_1(\mu)\,\hat{R}^{M-1} + \cdots + (-1)^{M-1}\,\sigma_{M-1}(\mu)\,\hat{R} + (-1)^M\,\sigma_M(\mu)\,\mathbf{1} = 0, \tag{3.1.65}$$

where $\mathbf{1}$ is a unit matrix in $V^{\otimes 2}$ and

$$\sigma_k(\mu) = \sum_{i_1 < i_2 < \cdots < i_k} \mu_{i_1} \ldots \mu_{i_k}$$

are elementary symmetric polynomials of $\mu_i$ ($i = 1, \dots, M$). For $R$-matrices satisfying (3.1.64), one can introduce a complete set of $M$ projectors:

$$\mathbf{P}_k = \prod_{j \neq k} \frac{(\hat{R} - \mu_j)}{(\mu_k - \mu_j)}, \qquad \sum_k \mathbf{P}_k = \mathbf{1}, \tag{3.1.66}$$

which project the $\hat{R}$-matrix to its eigenvalues $\mathbf{P}_k \, \hat{R} = \hat{R} \, \mathbf{P}_k = \mu_k \, \mathbf{P}_k$ and can be used for the spectral decomposition of an arbitrary function $X$ of $R$:

$$X(\hat{R}) = \sum_{k=1}^{M} X(\mu_k) \mathbf{P}_k. \tag{3.1.67}$$

In particular, for $X = \mathbf{1}$ we obtain the completeness condition (see the second equation in (3.1.66)). The derivation of formulas (3.1.66) can be found, for example, in [139, 180].

In the calculations of the knot/link invariants (3.1.57), the characteristic equations (3.1.65) play the role of the skein relations. We also note that for many known explicit examples of completely invertible Yang–Baxter $\hat{R}$-matrices, which satisfy the characteristic equation (3.1.64), all matrices $Y(X)$, defined in (3.1.43), are proportional to the identity matrix (see Proposition 4 in [47]).

**Examples.** Here we consider two special cases $M = 2, 3$ for the characteristic equation (3.1.64). By renormalizing the matrix $\hat{R}$, it is always possible to fix first two eigenvalues in (3.1.64) so that $\mu_1 = q \neq 0$ and $\mu_2 = -q^{-1} \neq 0$, where $q \in \mathbb{C}$.

**1.** For $M = 2$, Eqs. (3.1.64) and (3.1.65) are represented in the form of the Hecke condition

$$\begin{gathered} (\hat{R} - q)(\hat{R} + q^{-1}) = 0 \;\; \Rightarrow \;\; \hat{R}^2 = \lambda \, \hat{R} + \mathbf{1} \;\; \Rightarrow \\ \hat{R} - \lambda \, \mathbf{1} - \hat{R}^{-1} = 0, \quad \lambda := (q - q^{-1}). \end{gathered} \tag{3.1.68}$$

In this case, for all $n \in \mathbb{Z}$ we obtain

$$\hat{R}^n = \alpha_n \hat{R} + \alpha_{n-1} \mathbf{1}, \qquad \alpha_n := \frac{q^n - (-q)^{-n}}{q + q^{-1}}, \tag{3.1.69}$$

and according to (3.1.22), all matrices $Y(\hat{R}^n) \equiv Y^{(n)}$ in (3.1.46) are proportional to the identity matrix

$$\mathrm{Tr}_2(D_2 \hat{R}^n_{12}) = (\alpha_n + \alpha_{n-1} \mathrm{Tr}(D)) \, I_1. \tag{3.1.70}$$

In particular, one can immediately find (see (3.1.22), (3.1.28))

$$\mathrm{Tr}(Q) = \mathrm{Tr}(D), \qquad Y_1(\hat{R}^{-1}) = Q_1 \, D_1 = \mathrm{Tr}_2(D_2 \hat{R}^{-1}_1) = (1 - \lambda \, \mathrm{Tr}(D)) \; I_1 = q^{-2d} \, I_1, \tag{3.1.71}$$

where we introduce useful parametrization $q^{-2d} = (1 - \lambda \, \mathrm{Tr}(D))$. Equation (3.1.71) means that for the skew-invertible Hecke $R$-matrix, in the case $\lambda \mathrm{Tr}(D) \neq 1$, the matrices $D$ and $Q$ are always invertible and $Q^{-1} = q^{2d} \, D$.

**2.** For $M = 3$, Eqs. (3.1.64) and (3.1.65) are the Birman–Murakami–Wenzl cubic relations (cf. Eq. (3.10.4) below)

$$\begin{gathered} (\hat{R} - q)(\hat{R} + q^{-1})(\hat{R} - \nu) = 0 \;\; \Rightarrow \;\; \hat{R}^3 - (\lambda + \nu) \, \hat{R}^2 + (\lambda\nu - 1)\hat{R} + \nu \mathbf{1} = 0 \;\; \Rightarrow \\ \hat{K}\hat{R} = \hat{R}\hat{K} = \nu \hat{K}, \quad \hat{K} := \tfrac{1}{\lambda\nu}(q - \hat{R})(q^{-1} + \hat{R}) = \tfrac{1}{\lambda\nu}\Big(\mathbf{1} + \lambda\hat{R} - \hat{R}^2\Big), \end{gathered} \tag{3.1.72}$$

where $\lambda = (q - q^{-1})$. In this case, we have

$$\hat{K}^2 = \mu\,\hat{K}, \qquad \mu := \frac{1}{\lambda}(\nu^{-1} + \lambda - \nu), \tag{3.1.73}$$

and for all $n \in \mathbb{Z}$ we obtain

$$\hat{R}^n = \alpha_n \hat{R} + \alpha_{n-1}\mathbf{1} + \beta_n \hat{K}, \tag{3.1.74}$$
$$\alpha_n := \frac{q^n - (-q)^{-n}}{q + q^{-1}}, \quad \beta_n := \frac{\lambda\nu}{q+q^{-1}}\left(\frac{(\nu^n - (-q)^{-n})}{(\nu + q^{-1})} - \frac{(\nu^n - q^n)}{(\nu - q)}\right).$$

Let the matrix $\hat{K}$ be a one-dimensional projector in $V^{\otimes 2}$, i.e., $\hat{K}^{i_1 i_2}_{k_1 k_2} = \bar{C}^{i_1 i_2} C_{k_1 k_2}$. In this case, one can define the quantum trace (3.1.43) as follows (see Eq. (3.10.39) in Subsection **3.10** below):

$$\hat{K}_{23}\, X(\hat{R}_{12})\, \hat{K}_{23} = \nu^{-1}\mathrm{Tr}_2\big(X(\hat{R}_{12})D_2\big)\ \hat{K}_{23}, \qquad D^i_{\ j} := \nu\,\bar{C}^{ik}C_{jk},$$

and we deduce

$$\hat{K}_{23}\,\hat{R}_{12}\,\hat{K}_{23} = \nu^{-1}\hat{K}_{23}, \quad \hat{K}_{23}\,\hat{K}_{12}\,\hat{K}_{23} = \hat{K}_{23},$$
$$\hat{K}_{23}\,\mathbf{1}\,\hat{K}_{23} = \tfrac{1}{\lambda}(\nu^{-1} + \lambda - \nu)\,\hat{K}_{23} = \nu^{-1}\mathrm{Tr}(D)\hat{K}_{23} \quad \Rightarrow \quad \mathrm{Tr}(D) = \tfrac{(q-\nu)(q^{-1}+\nu)}{\lambda}.$$

Using these relations and (3.1.74), we obtain

$$\mathrm{Tr}_2(\hat{R}^n_{12}D_2) = \left(\alpha_n + \frac{(q-\nu)(q^{-1}+\nu)}{\lambda}\,\alpha_{n-1} + \nu\beta_n\right) I_1, \tag{3.1.75}$$

where $\alpha_n$ and $\beta_n$ were introduced in (3.1.74). Thus, for the cubic characteristic equation (3.1.72) all matrices $Y(\hat{R}^n)$ (3.1.46) are also proportional to the identity matrix and for $n = -1$ we find $Q^{-1} = \nu^{-2}\,D$.

**Remark 3.** Equations (3.1.70) and (3.1.75) are visualized in Figure 1A and give (for the cases $M = 2, 3$) invariants of links and knots (3.1.57):

$$M = 2: \quad \mathsf{Q}(\hat{R}^n_{12}) = \mathrm{Tr}_{12}(\hat{R}^n_{12}D_1D_2) = \big(\alpha_n + \alpha_{n-1}\mathrm{Tr}(D)\big)\,\mathrm{Tr}(D),$$
$$M = 3: \quad \mathsf{Q}(\hat{R}^n_{12}) = \mathrm{Tr}_{12}(\hat{R}^n_{12}D_1D_2) = \left(\alpha_n + \tfrac{(q-\nu)(q^{-1}+\nu)}{\lambda}\,\alpha_{n-1} + \nu\beta_n\right)\tfrac{(q-\nu)(q^{-1}+\nu)}{\lambda},$$

which are presented in Figure 1B:

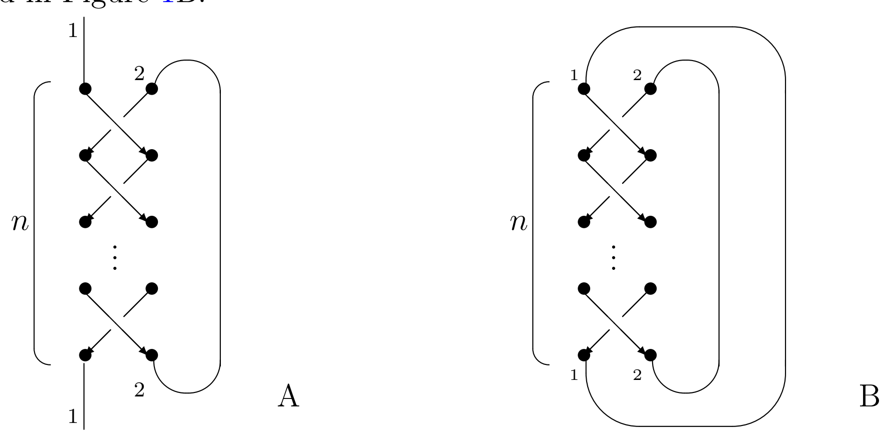



**Figure 1.** Closure of the braid $\hat{R}^n$ (the right picture B) gives toroidal knots for odd $n$ and links for even $n$.

The explicit examples of $R$-matrices subject to (3.1.68) and (3.1.72) with fixed values $\mathrm{Tr}(D)$ and $\nu$ are given in Subsections **3.4**, **3.7** and **3.10.1**, **3.11.2**, **3.11.3** below.

### *3.2. Quantum matrix algebras*

#### *3.2.1. RTT algebras*

We consider an algebra $\mathcal{A}^*$ of functions on a quasitriangular Hopf algebra $\mathcal{A}$ and assume that generators of $\mathcal{A}^*$ are the identity element 1 and elements of $N \times N$ matrix $T = ||T^i_j||$ $(i, j = 1, \dots, N)$, which define $N$-dimensional matrix representation of $\mathcal{A}$. We will use the following notation: $f(a) = \langle a, f\rangle$ for the functions $f \in \mathcal{A}^*$ of elements $a \in \mathcal{A}$. For the image $R_{12} = \langle \mathcal{R}, T_1 \otimes T_2\rangle$ of the universal matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ we deduce $\forall a \in \mathcal{A}$ the identity (by using (2.1.11) and (2.2.8))

$$R_{12}\,\langle a, T_1\,T_2\rangle = R_{12}\,\langle a_{(1)}, T_1\rangle\,\langle a_{(2)}, T_2\rangle = \langle\,\mathcal{R}\,\Delta(a), T_1 \otimes T_2\,\rangle =$$

$$= \langle\,\Delta^{\mathsf{op}}(a)\,\mathcal{R}, T_1 \otimes T_2\,\rangle = \langle\,\Delta^{\mathsf{op}}(a), T_1 \otimes T_2\,\rangle\,R_{12} = \langle a, T_2\,T_1\rangle\,R_{12}.$$

Since the element $a \in \mathcal{A}$ is not fixed here, one can conclude (for the nondegenerate pairing) that the elements $T^i_j$ satisfy the following quadratic relations ($RTT$ relations):

$$R^{i_1 i_2}_{j_1 j_2}\,T^{j_1}_{k_1}\,T^{j_2}_{k_2} = T^{i_2}_{j_2}\,T^{i_1}_{j_1}\,R^{j_1 j_2}_{k_1 k_2} \Leftrightarrow R_{12}T_1T_2 = T_2T_1R_{12} \Leftrightarrow \hat{R}\,T_1\,T_2 = T_1\,T_2\,\hat{R}, \tag{3.2.1}$$

where the indices 1 and 2 label the matrix spaces and the matrix $R_{12}$ satisfies Yang–Baxter equations (3.1.2), (3.1.5).

In the case of nontrivial $R$-matrices satisfying (3.1.2), the relations (3.2.1) define a noncommutative quadratic algebra (as the algebra of functions with the generators $\{1, T^i_j\}$), which is called the $RTT$ algebra. We stress that one can consider the $RTT$ algebra (3.2.1) with the Yang–Baxter $R$-matrix which is not in general the image of any universal $\mathcal{R}$-matrix. The Yang–Baxter equation for $R$ is necessary to ensure that on monomials of the third degree in $T$ no relations additional to (3.2.1) arise. We shall assume that $R_{12}$ is a skew-invertible matrix. In this case, matrices $D$ and $Q$ (3.1.20) define 1-dimensional representations $\rho_D(T^i_j) = D^i_j$ and $\rho_Q(T^i_j) = Q^i_j$ for the $RTT$ algebra (3.2.1) (see (3.1.36)). In some cases below, we also assume that $R_{12}$ is a lower triangular block matrix and its elements depend on the numerical parameter $q = \exp(h)$, which is called the deformation parameter.

Suppose that the $RTT$ algebra can be extended in such a way that it also contains all elements $(T^{-1})^i_j$:

$$(T^{-1})^i_k\,T^k_j = T^i_k\,(T^{-1})^k_j = \delta^i_j \cdot 1. \tag{3.2.2}$$

Then this algebra becomes a Hopf algebra with structure mappings

$$\Delta(T^i_k) = T^i_j \otimes T^j_k, \;\; \epsilon(T^i_j) = \delta^i_j, \;\; S(T^i_j) = (T^{-1})^i_j, \tag{3.2.3}$$

which, as is readily verified, satisfy the standard axioms (see Subsections **2.2** and **2.3**):

$$\begin{array}{c} (id \otimes \Delta)\Delta(T^i_j) = (\Delta \otimes id)\Delta(T^i_j), \\ (\epsilon \otimes id)\Delta(T^i_j) = (id \otimes \epsilon)\Delta(T^i_j) = T^i_j, \\ m(S \otimes id)\Delta(T^i_j) = m(id \otimes S)\Delta(T^i_j) = \epsilon(T^i_j)1. \end{array} \tag{3.2.4}$$

The antipode $S$ is not an involution, since instead of $S^2 = id$, we have an equation

$$S^2(T^i_j)\,D^j_l = D^i_k\,T^k_l, \tag{3.2.5}$$

which can be rewritten in the form

$$D^j_l T^l_k S(T^i_j) = D^i_k, \tag{3.2.6}$$

and the matrix $D$ has been defined in (3.1.20). The relations (3.2.5) and (3.2.6) can be interpreted as the rules of permutation of the operations of taking the inverse matrix and the transposition ($t$):

$$D^t (T^{-1})^t = (T^t)^{-1} D^t. \tag{3.2.7}$$

To prove (3.2.5), we note that $RTT$ relations (3.2.1) can be represented in the form

$$T_1^{-1}\,\hat{R}_{12}\,T_1 = T_2\,\hat{R}_{12}\,T_2^{-1}.$$

We multiply this relation by $\hat{\Psi}_{01}$ from the left and by $\hat{\Psi}_{23}$ from the right and take the traces $\mathrm{Tr}_{12}(\dots)$. Then, taking into account Eqs. (3.1.18), we arrive at the relation

$$\mathrm{Tr}_1\left(\hat{\Psi}_{01}\,T_1^{-1}\,P_{13}\,T_1\right) = \mathrm{Tr}_2\left(T_2\,P_{02}\,T_2^{-1}\,\hat{\Psi}_{23}\right).$$

Acting to this relation by $\mathrm{Tr}_3(\dots)$ and $\mathrm{Tr}_0(\dots)$, we obtain, respectively,

$$D_0 = \mathrm{Tr}_2 T_2\,P_{02}\,T_2^{-1}\,D_2, \quad \mathrm{Tr}_1 Q_1\,T_1^{-1}\,P_{13}\,T_1 \; = Q_3. \tag{3.2.8}$$

The first equation in (3.2.8) is identical to (3.2.5) and (3.2.6), while the second one gives

$$S(T^j_i)\,T^k_l\,Q^l_j = Q^k_i. \tag{3.2.9}$$

As it was shown in Subsection **3.1.2**, the matrices $D^i_j$ and $Q^i_j$ (3.1.20), entering the conditions (3.2.8), define *the quantum traces* [42, 67]. To explain the features of the quantum trace, we consider the $N^2$-dimensional adjoint $\mathcal{A}^*$-comodule $E$ (in what follows, we continue to use the concise notation $\mathcal{A}^*$ for the $RTT$ algebra). We represent its basis elements in the form of an $N \times N$ matrix $E = ||E^i_j||$, $i,j = 1,\dots,N$. The adjoint coaction is

$$E^i_j \to T^i_{i'} S(T^{j'}_j) \otimes E^{i'}_{j'} =: (TET^{-1})^i_j, \tag{3.2.10}$$

where in the right-hand side of (3.2.10), we have introduced abbreviations that we shall use in what follows (we omit the sign of the tensor product and should only remember that the elements $E^i_j$ commute with the elements $T^k_m$). We stress that there is a different form of the adjoint coaction:

$$E^i_j \to E^{i'}_{j'} \otimes S(T^i_{i'}) T^{j'}_j =: (T^{-1}ET)^i_j. \tag{3.2.11}$$

One can check that in (3.2.10) and (3.2.11), the elements $E^i_j$ form, respectively, left and right comodules. The matrix $||T^i_j||$ is interpreted now as the matrix of linear noncommutative adjoint transformations. Both left and right comodules $E$ are reducible, and irreducible subspaces in $E$ can be distinguished by means of the quantum traces. For the case (3.2.10), the quantum trace has the form (cf. (3.1.39))

$$\mathrm{Tr}_D E := \mathrm{Tr}(DE) \equiv \sum_{i,j=1}^{N} D^i_j E^j_i \tag{3.2.12}$$

and satisfies the following invariance property, which follows from Eqs. (3.2.5), (3.2.6) and the first relation in (3.2.8):

$$\mathrm{Tr}_D(TET^{-1}) = \mathrm{Tr}_D(E). \tag{3.2.13}$$

For the case (3.2.11), the definition of the quantum trace must be changed to

$$\mathrm{Tr}_Q E := \mathrm{Tr}(Q\,E) \equiv \sum_{i,j=1}^{N} Q^i_j\, E^j_i , \quad \mathrm{Tr}_Q(T^{-1} E T) = \mathrm{Tr}_Q(E), \tag{3.2.14}$$

this follows from the second relation in (3.2.8). Thus, $\mathrm{Tr}_D(E)$ and $\mathrm{Tr}_Q(E)$ are, respectively, the scalar parts of the comodules $E$ (3.2.10) and (3.2.11), whereas the $q$-traceless part of $E$ generates $(N^2-1)$-dimensional (reducible in the general case and irreducible in the case of linear quantum groups) $\mathcal{A}^*$-adjoint comodules. Note that, if the matrix $D$ is invertible, one can substitute $Q \to \mathrm{const}\cdot D^{-1}$ in (3.2.14), since Eq. (3.2.6) is rewritten in the form $(D^{-1})^k_i = S(T^j_i)\, T^k_l\, (D^{-1})^l_j$ (cf. (3.2.9)). We also note that formulas (3.2.13) and (3.2.14) of the adjoint invariancy of the quantum traces can be considered as universal analogs of (3.1.38).

An important consequence of the definition of the quantum trace (3.1.39), (3.2.13), (3.2.14) and $RTT$ relations (3.2.1) is the fact that

$$\begin{gathered} T_1^{-1}\, X(\hat{R})\, T_1 = T_2\, X(\hat{R})\, T_2^{-1} \quad \Rightarrow \\ T_1^{-1}\mathrm{Tr}_{D(2)}(X(\hat{R}))T_1 = \mathrm{Tr}_{D(2)}(X(\hat{R})), \quad T_2\, \mathrm{Tr}_{Q(1)}(X(\hat{R}))\, T_2^{-1} = \mathrm{Tr}_{Q(1)}(X(\hat{R})), \end{gathered} \tag{3.2.15}$$

where $X(.)$ is an arbitrary function, while $\mathrm{Tr}_{Q(1)}$ and $\mathrm{Tr}_{D(2)}$ are the quantum traces over the first and second space, respectively. Equation (3.2.15) indicates that the matrices $Y_2 = \mathrm{Tr}_{D(3)}(X(\hat{R}_2)) = \mathrm{Tr}_{Q(1)}(X(\hat{R}_1))$ (see (3.1.43)) must be proportional to the identity matrix if $T^i_j$ are functions which define an irreducible representation of the quantum group $\mathcal{A}$. In particular, we must have

$$\mathrm{Tr}_{D(3)}(\hat{R}^k_{23}) = \mathrm{Tr}_{Q(1)}(\hat{R}^k_{12}) = c_k I_2, \tag{3.2.16}$$

where $c_k$ are certain constants, e.g., $c_1 = 1$ (3.1.22), (3.1.23), and $I_k$ is the identity matrix in the $k$th space. Note that a direct consequence of (3.1.28) is

$$\mathrm{Tr}_{D(2)}(\hat{R}^{-1}_{12}) = c_{-1}\cdot I_1 = D_1\, Q_1, \tag{3.2.17}$$

and for $c_{-1} \neq 0$ matrices $D$, $Q$ are invertible. As we will see below, for the quantum groups of the classical series the fact (3.2.16) does indeed hold. In what follows, we shall attempt to restrict consideration to either left or right adjoint comodules with quantum traces (3.2.12) or (3.2.14). The analogous relations for right (or left) comodules can be considered exactly in the same way.

### *3.2.2. Faddeev–Reshetikhin–Takhtajan $L^{\pm}$ algebras*

It can be seen from comparison of the relations (3.2.1) and (3.1.2), (3.1.5) that for the generators $T^i_j$ it is possible to choose the following finite-dimensional matrix representations:

$$(T^i_j)^k_l = R^{ik}_{jl}, \quad (T^i_j)^k_l = (R^{-1})^{ki}_{lj}. \tag{3.2.18}$$

In these representations, the images of invariance relations (3.2.13), (3.2.14) coincide with (3.1.38). Since the $R$-matrix satisfies the Yang–Baxter equation, there exist linear functionals $(L^{\pm})^i_j \in \mathcal{A}$ that realize the homomorphisms (3.2.18), i.e., we have

$$\langle L_2^+, T_1\rangle = R_{12} := R^{(+)}_{12}, \;\; \langle L_2^-, T_1\rangle = R^{-1}_{21} := R^{(-)}_{12}. \tag{3.2.19}$$

For the case $R_{12} = \langle \mathcal{R}, T_1 \otimes T_2 \rangle$ we immediately construct the mapping from $\mathcal{A}^*$ to $\mathcal{A}$ (see, for example, [67, 91])

$$\langle \mathcal{R},\, id \otimes T^i_j \rangle = (L^+)^i_j, \quad \langle \mathcal{R},\, S(T^i_j) \otimes id \rangle = (L^-)^i_j, \\ \langle \mathcal{R},\, T^i_j \otimes id \rangle = S((L^-)^i_j). \tag{3.2.20}$$

Equations (3.2.19) are generalized in the following form:

$$\langle L^{\pm}_f, T_1\, T_2 \dots T_k \rangle = R^{(\pm)}_{1f}\, R^{(\pm)}_{2f} \dots R^{(\pm)}_{kf}.$$

The Yang–Baxter equation (3.1.5) can now be reproduced from $RTT$ relations (3.2.1) by averaging them with the $L^{\pm}$ operators.

From the requirement that elements $(L^{\pm})^i_j \in \mathcal{A}$ generate the algebra that is the dual to the algebra $\mathcal{A}^*$ (the definition of the dual algebra is given in Definition 6, Subsection **2.3**), we obtain the following commutation relations for the generators $L^{(\pm)}$:

$$\hat{R}_{12}\, L^{\pm}_2\, L^{\pm}_1 = L^{\pm}_2\, L^{\pm}_1\, \hat{R}_{12}, \tag{3.2.21}$$

$$\hat{R}_{12}\, L^{+}_2\, L^{-}_1 = L^{-}_2\, L^{+}_1\, \hat{R}_{12}. \tag{3.2.22}$$

The same equations are obtained from the universal Yang–Baxter equation (2.3.12) by the averaging it with $(T_1 \otimes T_2 \otimes id)$, $(id \otimes T_1 \otimes T_2)$, $(T_1 \otimes id \otimes T_2)$ and using (3.2.20). The algebra (3.2.21), (3.2.22) is obviously a Hopf algebra with comultiplication, antipode, and coidentity:

$$\triangle (L^{\pm})^i_j = (L^{\pm})^i_k \otimes (L^{\pm})^k_j, \;\; S(L^{\pm}) = (L^{\pm})^{-1}, \tag{3.2.23}$$

$$\epsilon((L^{\pm})^i_j) = \langle (L^{\pm})^i_j, 1 \rangle = \delta^i_j, \tag{3.2.24}$$

where we have assumed that the matrices $L^{\pm}$ are invertible.

We call the Hopf algebras with generators $\{(L^{\pm})^i_j\}$, defining relations (3.2.21), (3.2.22) and structure mappings (3.2.23), (3.2.24) as *Faddeev–Reshetikhin–Takhtajan (FRT) algebras*. As was shown in [42], for the $R$-matrices of the quantum groups of the classical series $A_n$, $B_n$, $C_n$, $D_n$ (respectively, $SL_q(n+1)$, $SO_q(2n+1)$, $Sp_q(2n)$, $SO_q(2n)$), the relations (3.2.21), (3.2.22) define quantum universal enveloping Lie algebras $U_q(sl(n+1))$, $U_q(so(2n+1))$, $U_q(sp(2n))$, $U_q(so(2n))$ in which the elements $(L^{\pm})^i_j$ play the role of the quantum analog of the Cartan–Weyl generators. We will investigate the case of $U_q(sl(n))$ below in Subsection **3.4**.

One can construct (see, e.g., [69]) the FRT algebra (3.2.21) – (3.2.24) as a Drinfeld double of two dual Hopf subalgebras $\mathcal{B}^+$ and $\mathcal{B}^-$ with generators $(L^+)^i_j$ and $(L^-)^i_j$, defining relations (3.2.21) and structure mappings (3.2.24), and (cf. (3.2.23))

$$\triangle (L^+)^i_j = (L^+)^i_k \otimes (L^+)^k_j, \quad \triangle_{\rm op}\, (L^-)^i_j = (L^-)^k_j \otimes (L^-)^i_k, \\ S(L^+) = (L^+)^{-1}, \;\; S^{-1}_{\rm op}(L^-) = (L^-)^{-1}. \tag{3.2.25}$$

In this case, the Hopf algebras $\mathcal{B}^+$ and $\mathcal{B}^-$ are dual to each other with respect to the pairing [69]:

$$\langle\!\langle L^-_1, L^+_2 \rangle\!\rangle = R^{-1}_{12}. \tag{3.2.26}$$

We denote by $\mathcal{B}^{-O}$ the Hopf algebra with generators $(L^-)^i_j$, and with comultiplication and antipode (3.2.23) opposite to that of (3.2.25). The algebras $\mathcal{B}^+$ and $\mathcal{B}^{-O}$ are antidual with respect to the pairing (3.2.26). As it was shown in Subsection **2.4**, from the antidual Hopf algebras $\mathcal{B}^+$ and $\mathcal{B}^{-O}$ it is possible to construct a Drinfeld quantum double $\mathcal{B}^+ \bowtie \mathcal{B}^{-O}$, for which the cross-commutation relations have the form (3.2.22). Thus, for the algebras $\mathcal{B}^\pm$ in (3.2.21) one can propose a special cross-product (quantum Drinfeld double), given by (3.2.22), which is again a Hopf algebra (with structure mappings (3.2.23), (3.2.24)), and which was used in [42] for the $R$-matrix formulation of quantum deformations of the universal enveloping Lie algebras.

Note that the FRT algebra (3.2.21), (3.2.22) is a covariant algebra (comodule algebra) with respect to the left and right cotransformations

$$\begin{gathered}(L^\pm)^i_j \to (T^{-1})^k_j \otimes (L^\pm)^i_k \equiv (L^\pm\, T^{-1})^i_j , \\ (L^\pm)^i_j \to (L^\pm)^k_j \otimes (T^{-1})^i_k \equiv (T^{-1}\, L^\pm)^i_j \end{gathered} \tag{3.2.27}$$

(we forget here for a moment that the matrices $T$ and $L^\pm$ could have the different triangular properties). Thus, the matrices

$$L^i_j = (S(L^-)L^+)^i_j, \quad \bar{L}^i_j = (L^+ S(L^-))^i_j \tag{3.2.28}$$

realize, respectively, the left and right adjoint comodules (3.2.10) and (3.2.11). It is clear that any powers $L^M$ and $\bar{L}^M$ are also the left and right adjoint comodules (3.2.10) and (3.2.11) and one can define the coinvariants

$$p_M = \mathrm{Tr}_D\left(L^M\right), \quad \overline{p}_M = \mathrm{Tr}_Q\left(\bar{L}^M\right). \tag{3.2.29}$$

**Proposition 3.6** *(see also [42]). The coinvariants (3.2.29) are central elements for the FRT algebra (3.2.21), (3.2.22) and $p_M = \overline{p}_M$ for the realizations (3.2.28).*

**Proof.** Indeed, one can obtain from (3.2.21), (3.2.22) the relations

$$L^M_2\, L^\pm_1 = L^\pm_1\, \hat{R}^{\pm 1}\, L^M_1\, \hat{R}^{\mp 1}, \quad L^\mp_2\, \bar{L}^M_1 = \hat{R}^{\pm 1}\, \bar{L}^M_2\, \hat{R}^{\mp 1}\, L^\mp_2, \tag{3.2.30}$$

where $\hat{R} := \hat{R}_{12}$. Then, by taking the traces $\mathrm{Tr}_{D(2)}$ and $\mathrm{Tr}_{Q(1)}$, respectively, of the first and second relations (3.2.30) and using (3.1.38), we prove $[p_M, L^\pm] = 0 = [\bar{p}_M, L^\pm]$, and therefore we demonstrate the centrality of the elements (3.2.29) for the algebra (3.2.21), (3.2.22).

The equality $p_M = \overline{p}_M$ for the elements (3.2.29) (where $L$ and $\overline{L}$ are composed from $L^\pm$ (3.2.28)) is deduced as follows:

$$\mathrm{Tr}_D\left(L^M\right) = \mathrm{Tr}_{D(2)}\left(S(L^-_2)\, \bar{L}^M_2\, L^-_2\right) = \mathrm{Tr}_{Q(1)}\mathrm{Tr}_{D(2)}\Big(S(L^-_2)\, \hat{R}\, \bar{L}^M_2\, L^-_2\Big) =$$

$$= \mathrm{Tr}_{Q(1)}\mathrm{Tr}_{D(2)}\Big(\bar{L}^M_1\, S(L^-_2)\, \hat{R}\, L^-_2\Big) = \mathrm{Tr}_{Q(1)}\mathrm{Tr}_{D(2)}\Big(\bar{L}^M_1\, L^-_1\, \hat{R}\, S(L^-_1)\Big) = \mathrm{Tr}_Q\left(\bar{L}^M\right),$$

where we have used Eqs. (3.1.22), (3.1.23), (3.2.21), (3.2.22), (3.2.30). ∎

*3.2.3. Reflection equation algebras*

Note also that the generators $L^i_j$ and $\overline{L}^i_j$ (3.2.28) satisfy the equations

$$\hat{R}_{12}\, L_1\, \hat{R}_{12}\, L_1 = L_1\, \hat{R}_{12}\, L_1\, \hat{R}_{12}, \tag{3.2.31}$$

$$\hat{R}_{12}\, \overline{L}_2\, \hat{R}_{12}\, \overline{L}_2 = \overline{L}_2\, \hat{R}_{12}\, \overline{L}_2\, \hat{R}_{12}. \tag{3.2.32}$$

In Subsection **5.2** below, we will see that (3.2.31) and (3.2.32) are the special limits of the reflection equations with spectral parameters. In view of this, algebras with generators $L^i_j$ and $\overline{L}^i_j$ and defining relations (3.2.31), (3.2.32) are called *the left and right reflection equation algebras*, since (3.2.31) and (3.2.32) are covariant under the left and right coactions (cotransformations) (3.2.10), (3.2.11). A set (which is incomplete in general; see below) of central elements for these algebras is represented by the same formulas as in (3.2.29). Indeed, one can deduce from (3.2.31), (3.2.32) the relations

$$L_1\,\hat{R}_{12}\,L_1^M\,\hat{R}_{12}^{-1} = \hat{R}_{12}^{-1}\,L_1^M\,\hat{R}_{12}\,L_1,\;\; \overline{L}_2\,\hat{R}_{12}\,\overline{L}_2^M\,\hat{R}_{12}^{-1} = \hat{R}_{12}^{-1}\,\overline{L}_2^M\,\hat{R}_{12}\,\overline{L}_2. \tag{3.2.33}$$

Then, taking the quantum traces $\mathrm{Tr}_{D(2)}(\dots)$ and $\mathrm{Tr}_{Q(1)}(\dots)$ of the first and second relations and using (3.1.38), we prove the centrality of the elements (3.2.29) for the algebras (3.2.31), (3.2.32)

$$[L^i_j,\ \mathrm{Tr}_D\left(L^M\right)] = 0, \qquad [\bar{L}^i_j,\ \mathrm{Tr}_Q\left(\bar{L}^M\right)] = 0. \tag{3.2.34}$$

The algebra (3.2.31) (and similarly the second algebra (3.2.32)) decomposes into the direct sum of two subalgebras, namely, into the Abelian algebra with generator $p_1 := \mathrm{Tr}_{_D}(L)$ and the algebra with $(N^2-1)$ traceless generators $\tilde{L}^i_j$ (we assume that $\mathrm{Tr}_{_D}(I) \neq 0$):

$$L^i_j = p_1'\,\delta^i_j + \lambda\,\tilde{L}^i_j \quad \Rightarrow \quad \tilde{L}^i_j = \frac{1}{\lambda}\left(L^i_j - p_1'\,\delta^i_j\right), \quad p_1' := \frac{p_1}{\mathrm{Tr}_{_D}(I)}, \tag{3.2.35}$$

where the factor $\lambda := q - q^{-1}$ is introduced to ensure that the operators $\tilde{L}$ have the correct classical limit for $q \to 1$. For the latest algebra, it is easy to obtain the commutation relations

$$\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12} = \frac{p_1'}{\lambda}(\tilde{L}_1\,\hat{R}_{12}^2 - \hat{R}_{12}^2\,\tilde{L}_1), \tag{3.2.36}$$

which after normalization $\tilde{L}_1 \to -p_1'\,\tilde{L}_1$ (for $p_1' \neq 0$) gives

$$\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12} = \frac{1}{\lambda}(\hat{R}_{12}^2\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}^2). \tag{3.2.37}$$

These relations can be regarded (for an arbitrary Yang–Baxter $R$-matrix) as a deformation of the commutation relations for Lie algebras. For the Hecke-type $R$-matrix (3.1.68) the relations (3.2.37) are equivalent to

$$\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12} = \hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}, \tag{3.2.38}$$

and corresponding algebra has a projector-type representation $\varrho$: $(\tilde{L}^i_j)^\alpha_\beta = A^{i\alpha}B_{j\beta}$, where numerical rectangular matrices $A$ and $B$ are such that $\mathrm{Tr}_\varrho(\tilde{L}^i_j) = B_{j\alpha}A^{i\alpha} = Q^i_j$ (for any matrix $Q$ that satisfies $\mathrm{Tr}_1 Q_1 \hat{R}_{12} = I_2$; see (3.1.22)).

The relations (3.2.31), (3.2.32), (3.2.37), and (3.2.38) are extremely important and arise, for example, in the construction of a differential calculus on quantum groups as the commutation relations for invariant vector fields (see [70–91] and references therein; see also Subsection **3.5.3** below).

Note that, instead of (3.2.20), one can use a somewhat different linear mapping from $\mathcal{A}^*$ to $\mathcal{A}$ [51, 67, 91, 95, 96] (which is completely determined by (3.2.20)):

$$\langle \sigma(\mathcal{R})\,\mathcal{R},\, id \otimes a\rangle = \alpha \quad (a \in \mathcal{A}^*, \alpha \in \mathcal{A}), \tag{3.2.39}$$

where $\sigma(a \otimes b) = (b \otimes a)$, $\forall a, b \in \mathcal{A}$. The explicit calculations give

$$\langle \sigma(\mathcal{R})\,\mathcal{R},\, id \otimes T^i_j \rangle = L^i_j, \tag{3.2.40}$$

$$\langle \sigma(\mathcal{R})\,\mathcal{R},\, id \otimes T_1\,T_2 \rangle = S(L^-_1)\,L_2\,L^+_1 = L_1\,\hat{R}_1\,L_1\,\hat{R}_1^{-1},$$

$$\langle \sigma(\mathcal{R})\,\mathcal{R},\, id \otimes T_1\,T_2\,T_3 \rangle = S(L^-_1)\,S(L^-_2)\,L_3\,L^+_2\,L^+_1 = L_{\underline{1}}\,L_{\underline{2}}\,L_{\underline{3}} \equiv L_{\overline{3}}\,L_{\overline{2}}\,L_{\overline{1}}, \tag{3.2.41}$$

$$\dots\dots\dots\dots,$$

$$\langle \sigma(\mathcal{R})\,\mathcal{R},\, id \otimes T_1 \dots T_k \rangle = L_{\underline{1}}\,L_{\underline{2}} \dots L_{\underline{k}} \equiv L_{\overline{k}} \dots L_{\overline{2}}\,L_{\overline{1}},$$

where

$$L_{\underline{k+1}} = \hat{R}_k\,L_{\underline{k}}\,\hat{R}_k^{-1}, \quad L_{\overline{k+1}} = \hat{R}_k^{-1}\,L_{\overline{k}}\,\hat{R}_k, \quad L_{\underline{1}} = L_{\overline{1}} = L_1, \tag{3.2.42}$$

and we have used Eqs. (2.3.9), (3.2.20), and (3.2.30). If we confine ourselves to the fairly general case of quasitriangular Hopf algebras $\mathcal{A}$, for which the mapping (3.2.39) is invertible (such Hopf algebras are called factorizable [95]), one can map the identities for the $RTT$ algebra into the identities for the reflection equation algebra and vice versa. For this we need to use relations (3.2.39) (for more details see [96]).

In view of (3.2.41), one can represent the reflection equation algebra (3.2.31) in the "universal" form

$$\mathcal{R}_{32}\,(\mathcal{R}_{31}\mathcal{R}_{13})\,\mathcal{R}_{23}\,(\mathcal{R}_{21}\,\mathcal{R}_{12}) = (\mathcal{R}_{21}\,\mathcal{R}_{12})\,\mathcal{R}_{32}\,(\mathcal{R}_{31}\mathcal{R}_{13})\,\mathcal{R}_{23},$$

where the notation $\mathcal{R}_{ij}$ has been introduced in (2.3.11). The pairing of this relation with $(id \otimes T \otimes T)$ gives (3.2.31). The algebra (3.2.32) has an analogous representation if we start with

$$\langle \sigma(\mathcal{R})\,\mathcal{R},\, T^i_j \otimes id \rangle = \overline{L}^i_j. \tag{3.2.43}$$

We note that the identity (which has been obtained in (3.2.41))

$$L_{\underline{1}}\,L_{\underline{2}} \dots L_{\underline{k}} = L_{\overline{k}} \dots L_{\overline{2}}\,L_{\overline{1}} \tag{3.2.44}$$

is valid in more general case of any reflection equation algebra (3.2.31) (even not realized in the form (3.2.28)). Below we also use the following identity (which can be proved by induction):

$$L_{\underline{k+1}}\,L_{\underline{k+2}} \dots L_{\underline{k+n}} = U_{(k,n)}\,L_{\underline{1}}\,L_{\underline{2}} \dots L_{\underline{n}}\,U^{-1}_{(k,n)}, \tag{3.2.45}$$

where the operator $U_{(k,n)}$ is represented as a product of $k$ or $n$ factors (cf. (3.1.60)):

$$U_{(k,n)} = \hat{R}_{(k\to n+k-1)} \dots \hat{R}_{(2\to n+1)}\,\hat{R}_{(1\to n)} \equiv \hat{R}_{(k\leftarrow 1)}\,\hat{R}_{(k+1\leftarrow 2)} \dots \hat{R}_{(n+k-1\leftarrow n)}, \tag{3.2.46}$$

$$\hat{R}_{(k\leftarrow m)} := \hat{R}_k\,\hat{R}_{k-1} \cdots \hat{R}_m, \quad \hat{R}_{(m\to k)} := \hat{R}_m\,\hat{R}_{m+1} \cdots \hat{R}_k. \tag{3.2.47}$$

### *3.2.4. Central and commuting subalgebras for reflection equation and RTT algebras*

As we prove in the previous subsection, the elements (3.2.29) are central for the $RLRL$ (reflection equation) algebras (3.2.31), (3.2.32). Now the description of a more general set of central elements for reflection equation algebra is in order.

**Proposition 3.7.** *Let $X_{(1\to m)}$ be an arbitrary element of the group algebra of the braid group $\mathcal{B}_m$ generated by skew-invertible R-matrices $\hat{R}_a$ $(a = 1, \dots, m-1)$ with defining relations (3.1.11), (3.1.10), (3.1.12). Then the elements*

$$z_m(X) = \mathrm{Tr}_{\mathcal{D}(1\dots m)}\left(X_{(1\to m)}\,L_{\underline{1}}\,L_{\underline{2}} \dots L_{\underline{m}}\right) \quad (m = 1, 2, \dots) \tag{3.2.48}$$

*belong to the center $Z(L)$ of the reflection equation algebra (3.2.31), where we recall $\hat{R}_{12} \equiv \hat{R}_1$.*

**Proof.** First of all, we note that $z_m(X)$ (3.2.48) satisfies

$$\begin{array}{c} z_m(X)\, I_1 = \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( X_{(2\to m+1)}\, L_{\underline{2}}\, L_{\underline{3}} \ldots L_{\underline{m+1}} \right) = \\ = \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( X_{(2\to m+1)}\, L_{\overline{m+1}} \ldots L_{\overline{3}}\, L_{\overline{2}} \right), \end{array} \tag{3.2.49}$$

where $X_{(2\to m+1)} \in \mathcal{B}_{m+1}$ is obtained from $X_{(1\to m)}$ by the shift $R_a \to R_{a+1}$ $(\forall a)$. The first equality follows from the chain of relations

$$\begin{array}{c} \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( X_{(2\to m+1)}\, L_{\underline{2}} \ldots L_{\underline{m+1}} \right) = \\ = \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( X_{(2\to m+1)}\, \hat{R}_1 \cdots \hat{R}_m\, L_{\underline{1}} \ldots L_{\underline{m}}\, \hat{R}_m^{-1} \cdots \hat{R}_1^{-1} \right) = \\ = \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( \hat{R}_1 \cdots \hat{R}_m\, (X_{(1\to m)}\, L_{\underline{1}} \ldots L_{\underline{m}})\, \hat{R}_m^{-1} \cdots \hat{R}_1^{-1} \right) = \\ = \mathrm{Tr}_{\mathcal{D}(2\ldots m)} \left( \hat{R}_1 \cdots \hat{R}_{m-1}\, [\mathrm{Tr}_{\mathcal{D}(m)}(X_{(1\to m)}\, L_{\underline{1}} \ldots L_{\underline{m}})]\, \hat{R}_{m-1}^{-1} \cdots \hat{R}_1^{-1} \right) = \\ = \cdots = I_1\, \mathrm{Tr}_{\mathcal{D}(1\ldots m)} \left( X_{(1\to m)}\, L_{\underline{1}}\, L_{\underline{2}} \ldots L_{\underline{m}} \right), \end{array}$$

where we have applied (3.1.38) many times. The second equality in (3.2.49) is proved in the same way, or by using the generalization of the identity (3.2.44)

$$L_{\underline{m}}\, L_{\underline{m+1}} \ldots L_{\underline{k}} = L_{\overline{k}} \ldots L_{\overline{m+1}}\, L_{\overline{m}} \quad (m < k).$$

Then the proof of the commutativity of the arbitrary generator of the reflection equation (RE) algebra (3.2.31) with elements $z_m(X)$ is straightforward:

$$\begin{array}{c} L_1\, z_m(X) = \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( X_{(2\to m+1)}\, L_{\underline{1}} L_{\underline{2}}\, L_{\underline{3}} \ldots L_{\underline{m+1}} \right) = \\ = \mathrm{Tr}_{\mathcal{D}(2\ldots m+1)} \left( X_{(2\to m+1)}\, L_{\overline{m+1}} \ldots L_{\overline{2}}\, L_{\overline{1}} \right) = z_m(X)\, L_1. \end{array}$$

■

**Remark 1.** If, in the definition of central generators (3.2.48), we take the set of elements $X = X_\alpha$, $\alpha = 1, 2, \ldots$, which are all primitive idempotents for any finite-dimensional quotient $\mathcal{B}'_m$ of the group algebra of $\mathcal{B}_m$, then the set of central elements $z_m(X_\alpha)$ forms a basis in the subspace of $Z(L)$ generated by elements (3.2.48) for any matrices $X \in \mathcal{B}'_m$.

**Remark 2.** The "power sums" (3.2.29) belong to the space $Z(L)$. Indeed, the substitution of $X = \hat{R}_{(m-1\leftarrow 1)} := \hat{R}_{m-1} \ldots \hat{R}_1$ in (3.2.48) gives

$$\begin{array}{c} z_m(X) = \mathrm{Tr}_{\mathcal{D}(1\ldots m)} \left( L_{\underline{1}} \ldots L_{\underline{m-1}}\, (\hat{R}_{(m-1\leftarrow 1)} L_1 \hat{R}^{-1}_{(m-1\leftarrow 1)})\, \hat{R}_{(m-1\leftarrow 1)} \right) = \\ = \mathrm{Tr}_{\mathcal{D}(1\ldots m)} \left( L_{\underline{1}} \ldots L_{\underline{m-2}}\, (\hat{R}_{(m-2\leftarrow 1)} L_1 \hat{R}^{-1}_{(m-2\leftarrow 1)}) \hat{R}_{m-1} \hat{R}_{(m-2\leftarrow 1)} L_1 \right) = \\ = \mathrm{Tr}_{\mathcal{D}(1\ldots m-1)} \left( L_{\underline{1}} \ldots L_{\underline{m-2}}\, \hat{R}_{(m-2\leftarrow 1)} L_1^2 \right) = \cdots = \mathrm{Tr}_{D(1)}(L_1^m) = p_m, \end{array} \tag{3.2.50}$$

where in the first line we used the cyclic property of the quantum trace (3.1.40) and in the second line we applied (3.1.22).

Now we discuss the set of commuting elements in the $RTT$ algebra (3.2.1). For this algebra one can construct [94] the following elements:

$$Q_k = \mathrm{Tr}_{Y(1\ldots k)}(\hat{R}_{(k-1\leftarrow 1)}\, T_1\, T_2 \cdots T_k) = \mathrm{Tr}_{Y(1\ldots k)}(\hat{R}_{(1\to k-1)}\, T_1\, T_2 \cdots T_k), \tag{3.2.51}$$

where

$$\mathrm{Tr}_{Y(1\ldots k)}(X_{1\ldots k}) := \mathrm{Tr}_1 \ldots \mathrm{Tr}_k(Y_1 \ldots Y_k\, X_{1\ldots k}),$$

and the matrices $Y$ are such that $Y_1Y_2\hat{R}_1 = \hat{R}_1Y_1Y_2$ (e.g., $Y = D$ or $Y = Q$, see (3.1.36)). The second equality in (3.2.51) is obtained as follows:

$$\begin{array}{c} \mathrm{Tr}_{Y(1\dots k)}(\hat{R}_1\cdots\hat{R}_{k-1}\,T_1\cdots T_k) = \mathrm{Tr}_{Y(1\dots k)}(\hat{R}_{k-1}\,T_1\cdots T_k\,\hat{R}_1\cdots\hat{R}_{k-2}) = \\ = \mathrm{Tr}_{Y(1\dots k)}(\hat{R}_1\cdots\hat{R}_{k-3}\,\hat{R}_{k-1}\,\hat{R}_{k-2}\,T_1\cdots T_k\,) = \\ = \mathrm{Tr}_{Y(1\dots k)}(\hat{R}_{k-1}\,\hat{R}_{k-2}\,T_1\cdots T_k\,\hat{R}_1\cdots\hat{R}_{k-3}) = \\ = \cdots = \mathrm{Tr}_{Y(1\dots k)}(\hat{R}_{k-1}\,\hat{R}_{k-2}\cdots\hat{R}_1\,T_1\cdots T_k). \end{array} \tag{3.2.52}$$

Note that by means of (3.2.39) we map the elements $Q_k$ (3.2.51) (for $Y = D$) to the central elements $p_k$ (3.2.29) of the reflection equation algebra.

**Proposition 3.8.** *The elements (3.2.51) generate a commutative subalgebra in the RTT algebra (3.2.1).*

**Proof.** Our proof of the commutativity of the elements $Q_k$ is based (see [97]) on the fact that there exists the operator $U_{(k,n)}$ (3.2.46) which satisfies

$$U_{(k,n)}\,\hat{R}_i\,U^{-1}_{(k,n)} = \hat{R}_{i+k}, \;\; i = 1,\dots,n-1, \qquad U_{(k,n)}\,\hat{R}_{n+j}\,U^{-1}_{(k,n)} = \hat{R}_j, \;\; j = 1,\dots,k-1.$$

Using the operator $U_{(k,n)}$, we obtain the commutativity of $Q_k$:

$$\begin{array}{c} Q_k\,Q_n = \mathrm{Tr}_{Y(1\dots k)}(\hat{R}_{(1\to k-1)}\,T_1\cdots T_k)\,\mathrm{Tr}_{Y(1\dots n)}(\hat{R}_{(1\to n-1)}\,T_1\cdots T_n) = \\ = \mathrm{Tr}_{Y(1\dots k+n)}(\hat{R}_{(1\to k-1)}\,\hat{R}_{(k+1\to k+n-1)}\,T_1\cdots T_{k+n}) = \\ = \mathrm{Tr}_{Y(1\dots k+n)}(U_{(k,n)}\,\hat{R}_{(n+1\to n+k-1)}\,\hat{R}_{(1\to n-1)}\,U^{-1}_{(k,n)}\,T_1\cdots T_{k+n}) = \\ = \mathrm{Tr}_{Y(1\dots k+n)}(\hat{R}_{(1\to n-1)}\,\hat{R}_{(n+1\to n+k-1)}\,U^{-1}_{(k,n)}\,T_1\cdots T_{k+n}\,U_{(k,n)}) = Q_n\,Q_k. \end{array} \tag{3.2.53}$$

■

In fact, applying the same method as in (3.2.53), one can prove [97] that the set of commuting elements in the *RTT* algebra is wider then the set (3.2.51) and consists of all elements of the form

$$Q_k(X) = \mathrm{Tr}_{Y(1\dots k)}\left(X(\hat{R}_1,\dots\hat{R}_{k-1})\,T_1\,T_2\cdots T_k\right), \tag{3.2.54}$$

where $X(\dots)$ run over basis elements of the braid group algebra with generators $\{\hat{R}_i\}$ ($i = 1,\dots,k-1$).

Our conjecture is that, for the Hecke-type $R$-matrices (3.1.68), the set of elements (3.2.51)

$$Q_k := Q_k(\hat{R}_{1\to k-1}) \equiv Q_k(\hat{R}_{k-1\leftarrow 1})$$

is complete and all $Q_k(X)$ (for any braid $X$ with $k$ strands) are expressed as polynomials of the commuting variables $\{Q_1,\dots,Q_k\}$ and deformation parameter $q$. These polynomials, if we add some extra constraints dictated by Markov (Reidemeister) moves for the braids $X$ (see Section 1 in [201]), could be related to link polynomials. On the other hand, Eq. (3.2.54) defines $q$-analogs of characters for representations of the algebra $\mathcal{A}$ (3.2.21)–(3.2.24) and for the *RTT* algebra $\mathcal{A}^*$. These representations are characterized by special choices of the elements $X(\dots)$ being central idempotents in the Hecke algebra generated by matrices $\{\hat{R}_i\}$ ($i = 1,\dots,k-1$). We will discuss these ideas in detail in Subsection **4.3.6** below.

*3.2.5. Heisenberg double for the RTT and reflection equation algebras*

Since the $RTT$ algebra $\mathcal{A}^*$ (3.2.1), (3.2.3) and the quantum algebra $\mathcal{A}$ (3.2.21)–(3.2.24) are Hopf dual to each other (with respect to the pairing (3.2.19)), one can define the left and right Heisenberg doubles (HD) of these algebras (about HD see Subsection **2.4**). Their cross-multiplication rules (2.4.1), (2.4.2) are written for the left HD in the form

$$L_1^+ T_2 = T_2 R_{21} L_1^+, \quad L_1^- T_2 = T_2 R_{12}^{-1} L_1^-, \tag{3.2.55}$$

and for the right one we have

$$T_1 L_2^+ = L_2^+ R_{12} T_1, \quad T_1 L_2^- = L_2^- R_{21}^{-1} T_1. \tag{3.2.56}$$

The corresponding cross products of the $RTT$ algebra and the reflection equation algebras (3.2.28), (3.2.31), (3.2.32) are described by the cross-multiplication rules

$$\overline{L}_1 T_2 = T_2 \hat{R}_{12} \overline{L}_2 \hat{R}_{12}, \quad T_1 L_2 = \hat{R}_{12} L_1 \hat{R}_{12} T_1 \tag{3.2.57}$$

in the case of the left (3.2.55) and right (3.2.56) HD, respectively. A remarkable property [98] of these cross products is the existence of automorphisms of the HD algebras

$$\{T, \overline{L}\} \overset{\overline{m}_n}{\to} \{T\, \overline{L}^n, \overline{L}\}, \quad \{T, L\} \overset{m_n}{\to} \{L^n T, L\}, \tag{3.2.58}$$

i.e., we have (the same is valid for the automorphisms $\overline{m}_n$)

$$\hat{R}_{12} (L^n T)_1 (L^n T)_2 = (L^n T)_1 (L^n T)_2 \hat{R}_{12}, \qquad (L^n T)_1 L_2 = \hat{R}_{12} L_1 \hat{R}_{12} (L^n T)_1 \;\; \Rightarrow \tag{3.2.59}$$
$$(L^n T)_1 L_2^k = (\hat{R}_{12} L_1 \hat{R}_{12})^k (L^n T)_1, \qquad \forall n, k \in \mathbb{Z}_{\geqslant 0}.$$

One can check these properties by induction using Eqs. (3.2.1), (3.2.31), (3.2.32), and (3.2.57). The maps $m_n, \overline{m}_n$ define discrete time evolutions on the $RTT$ algebra. For the Hecke-type $R$-matrices (3.1.68) the automorphisms (3.2.58) can be generalized in the form

$$\{T, \overline{L}\} \overset{\overline{m}'_n}{\to} \left\{ T \left( \sum_{m=0}^{n} \bar{x}_m \overline{L}^m \right), \overline{L} \right\}, \quad \{T, L\} \overset{m'_n}{\to} \left\{ \left( \sum_{m=0}^{n} x_m L^m \right) T, L \right\}, \tag{3.2.60}$$

for any parameters $x_m, \bar{x}_m \in \mathbb{C}$. This generalization follows from the fact that any symmetric function of two variables $L_1$ and $\hat{R}_1 L_1 \hat{R}_1$ commutes with $\hat{R}_1$.

For the left and right Heisenberg doubles (3.2.55)–(3.2.57) one can define new reflection equation algebras, generated by the elements of matrices $L$ and $\overline{L}$ transformed by the adjoint action of the $RTT$ algebra

$$\overline{Y} = T\, \overline{L}^{-1} T^{-1}, \quad Y = T^{-1} L^{-1} T,$$

for which we have [84] (cf. (3.2.31), (3.2.32), (3.2.57)):

$$\hat{R}_{12} \overline{Y}_1 \hat{R}_{12} \overline{Y}_1 = \overline{Y}_1 \hat{R}_{12} \overline{Y}_1 \hat{R}_{12}, \quad \hat{R}_{12} Y_2 \hat{R}_{12} Y_2 = Y_2 \hat{R}_{12} Y_2 \hat{R}_{12},$$

$$T_1 \overline{Y}_2 = \hat{R}_{12} \overline{Y}_1 \hat{R}_{12} T_1, \quad Y_1 T_2 = T_2 \hat{R}_{12} Y_2 \hat{R}_{12}.$$

The elements of these matrices satisfy: $[\overline{Y}_2, \overline{L}_1] = 0 = [Y_1, L_2]$. In the differential calculus on quantum groups, matrices $L$ and $M := Y^{-1}$ are interpreted (see [84] and [129]) as invariant vector fields on the $RTT$ algebras (see Proposition **3.9** below).

The cross-multiplication rules (3.2.57) for the HD of the $RTT$ and reflection equation algebras were extensively exploited in the context of the $R$-matrix approach to the differential calculus on quantum groups [71–91] (see also Subsection **3.5.3** below). Another cross-multiplications (of the $RTT$ and reflection equation matrix algebras), which are characterized by the relations

$$\overline{L}_1 T_2 = T_2 \hat{R}_{12} \overline{L}_2 \hat{R}_{12}^{-1}, \qquad T_1 L_2 = \hat{R}_{12} L_1 \hat{R}_{12}^{-1} T_1, \tag{3.2.61}$$

were also considered in various investigations [84, 87, 88] of a noncommutative differential geometry on quantum groups.

**Proposition 3.9.** *1. For cross-multiplication of the RTT and reflection equation algebras (REA) with generators $T^i_j$ and $L^i_j$ subject to defining relations (see (3.2.1), (3.2.31), and (3.2.61))*

$$\hat{R}_{12}\, T_1 T_2 = T_1 T_2\, \hat{R}_{12}, \quad \hat{R}_{12} L_1\, \hat{R}_{12} L_1 = L_1\, \hat{R}_{12} L_1\, \hat{R}_{12}, \quad T_1 L_2 = \hat{R}_{12} L_1 \hat{R}_{12}^{-1} T_1, \tag{3.2.62}$$

*we have the following equations [88]:*

$$\hat{R}_{12}\, (L\,T)_1\, (L\,T)_2 = (L\,T)_1\, (L\,T)_2\, \hat{R}_{12} \tag{3.2.63}$$

*(we, however, stress that it is impossible to define the whole discrete evolution (3.2.59) for the double algebra (3.2.62)).*
*2. Let $L^i_j$, $\tilde{L}^i_j$ be generators of the REA (3.2.31) and $\tilde{L}^i_j$ subject to the following cross-commutation relations [88] with generators of (3.2.62)*

$$T_1 \tilde{L}_2 = \hat{R}_{12} \tilde{L}_1 \hat{R}_{12}^{-1} T_1, \quad \hat{R}_{12}^{-1} \tilde{L}_1\, \hat{R}_{12} L_1 = L_1\, \hat{R}_{12}^{-1} \tilde{L}_1\, \hat{R}_{12}. \tag{3.2.64}$$

*Then we have [52, 88]*

$$\begin{gathered} \hat{R}_{12}\, (\tilde{L}\,T)_1\, L_2 = \hat{R}_{12} L_1\, \hat{R}_{12}^{-1} (\tilde{L}\,T)_1, \qquad \hat{R}_{12}\, (\tilde{L}\,T)_1\, (\tilde{L}\,T)_2 = (\tilde{L}\,T)_1\, (\tilde{L}\,T)_2\, \hat{R}_{12}, \\ \hat{R}_{12} (L\,\tilde{L})_1\, \hat{R}_{12} (L\,\tilde{L})_1 = (L\,\tilde{L})_1\, \hat{R}_{12} (L\,\tilde{L})_1\, \hat{R}_{12}. \end{gathered} \tag{3.2.65}$$

*3.2.6. Quantum matrix algebras in general setting*

Now we present a definition of a more general quantum matrix algebra $\mathcal{M}(\hat{R}, \hat{F})$ generated by $(N \times N)$ matrix components $M^i_j$ subject to the relation

$$\hat{R}_{12}\, M_1\, \hat{F}_{12}\, M_1\, \hat{F}_{12} = M_1\, \hat{F}_{12}\, M_1\, \hat{F}_{12}\, \hat{R}_{12}, \tag{3.2.66}$$

where the pair of Yang–Baxter operators $\{\hat{R}, \hat{F}\} \in \mathrm{End}(V_N^{\otimes 2})$ satisfies the conditions

$$\hat{R}_{12}\, \hat{F}_{23}\, \hat{F}_{12} = \hat{F}_{23}\, \hat{F}_{12}\, \hat{R}_{23}, \quad \hat{R}_{23}\, \hat{F}_{12}\, \hat{F}_{23} = \hat{F}_{12}\, \hat{F}_{23}\, \hat{R}_{12}. \tag{3.2.67}$$

The algebra $\mathcal{M}(\hat{R}, \hat{F})$ is a quantum matrix algebra $\mathcal{M}(\sigma(\mathcal{R})\mathcal{F})$, since we can reproduce (3.2.66) (for details see [99]) by means of identifications

$$M^i_j := \langle \sigma(\mathcal{R})\mathcal{F}, id \otimes T^i_j \rangle, \quad \hat{F}_{12} := P_{12} \langle \mathcal{F},\, T_1 \otimes T_2 \rangle,$$

where $\mathcal{F}$ is a twisting matrix (2.5.4), (2.5.7) and $P_{12}$ is the permutation matrix (3.1.8). Note that Eqs. (3.2.67) are the images of Eqs. (2.5.8). It means that, for the pair the Yang–Baxter operators $\{\hat{R}, \hat{F}\}$ (3.2.67), the matrix

$$\hat{R}^F_{21} = \hat{F}_{12} \hat{R}_{12} \hat{F}_{12}^{-1} = \langle \mathcal{R}^{\mathcal{F}}, T_1 \otimes T_2 \rangle P_{12} \tag{3.2.68}$$

is the Yang–Baxter matrix as well. Specializing to $\hat{F} = P$ or $\hat{F} = \hat{R}$, one reproduces from (3.2.66) the $RTT$ or reflection equation algebras, respectively. The algebras $\mathcal{M}(\hat{R}, \hat{F})$ (3.2.66) and their modifications were discussed in [97, 99, 100, 130].

At the end of this subsection, we introduce a notion of a coideal subalgebra of the quantum algebra (3.2.21), (3.2.22). Let $R_{12}$ be a Yang–Baxter $R$-matrix and there are numerical matrices $G^i_j$, $\overline{G}^i_j$ which satisfy the conditions

$$\begin{aligned} R_{12}\, G_2\, R_{12}^{t_2}\, G_1 &= G_1\, R_{21}^{t_1}\, G_2\, R_{21}^{t_1 t_2}, \\ R_{12}\, \overline{G}_1\, R_{12}^{t_1}\, \overline{G}_2 &= \overline{G}_2\, R_{21}^{t_2}\, \overline{G}_1\, R_{21}^{t_1 t_2}. \end{aligned} \tag{3.2.69}$$

Using relations (3.2.21), (3.2.22) and conditions (3.2.69), it can be shown directly that the elements of quantum matrices

$$K = L^-\, G\, (L^+)^t, \quad \overline{K} = S(L^+)\, \overline{G}\, (S(L^-))^t$$

obey the following commutation relations:

$$\begin{aligned} R_{12}\, K_2\, R_{12}^{t_2}\, K_1 &= K_1\, R_{21}^{t_1}\, K_2\, R_{21}^{t_1 t_2}, \\ R_{12}\, \overline{K}_1\, R_{12}^{t_1}\, \overline{K}_2 &= \overline{K}_2\, R_{21}^{t_2}\, \overline{K}_1\, R_{21}^{t_1 t_2}, \end{aligned} \tag{3.2.70}$$

which we consider as the defining relations for a new type of quantum matrix algebras $\mathcal{K}(\hat{R}, G)$ and $\overline{\mathcal{K}}(\hat{R}, \overline{G})$. The defining relations (3.2.70) are covariant[8] under the left and right $\mathcal{A}$-coactions:

$$K^i_j \longrightarrow (L^-)^i_k\, (L^+)^j_n \otimes K^k_n, \quad \overline{K}^i_j \longrightarrow \overline{K}^k_n \otimes S(L^+)^i_k\, S(L^-)^j_n. \tag{3.2.71}$$

Thus, the unital algebras $\mathcal{K}$ and $\overline{\mathcal{K}}$ (with generators $K^i_j$ and $\overline{K}^i_j$, respectively) are left and right $\mathcal{A}$-comodule algebras and these algebras are called coideal subalgebras of $\mathcal{A}$.

One can consider two more such algebras with generators $K' = L^+ G' (L^-)^t$ and $\overline{K}' = S(L^-) \overline{G}' S(L^+)^t$ which obey the following defining relations:

$$\begin{aligned} R_{12}^{-1}\, K_1'\, (R_{12}^{-1})^{t_1}\, K_2' &= K_2'\, (R_{21}^{-1})^{t_2}\, K_1'\, (R_{21}^{-1})^{t_1 t_2}, \\ R_{12}^{-1}\, \overline{K}_2'\, (R_{12}^{-1})^{t_2}\, \overline{K}_1' &= \overline{K}_1'\, (R_{21}^{-1})^{t_1}\, \overline{K}_2'\, (R_{21}^{-1})^{t_1 t_2}. \end{aligned}$$

Note that these relations can be obtained from (3.2.70) by the substitution $R_{12} \to R_{12}^{-1}$.

For the special case of $GL_q(N)$ $R$-matrices (see Subsection **3.4**) the algebras (3.2.70) have been considered in [101, 106] (see also references therein). In this case, the coideal subalgebras coincide with quantized enveloping algebras introduced earlier by A. Gavrilik and A. Klimyk [105].

Representation theory for compact quantum groups has been considered in [116]. In [117], a universal solution to the reflection equation has been introduced and general problems of the representation theory for the reflection equation algebra were discussed (representations and characters for some special reflection equation algebras were considered in [24, 118, 119]). A classification of commutative solutions of the graded reflection equations associated with the vector representations of the quantum supergroup of GL-type was given in [120].

[8]Here the notion *covariant* is equivalent to the statement that (3.2.71) are homomorphisms for the algebras defined by (3.2.70).

*3.3. The semiclassical limit (Sklyanin brackets and Lie bialgebras)*

We assume that the $R$-matrix introduced in (3.2.1) has the following expansion in the limit $h \to 0$ ($q = e^h \to 1$):

$$R_{12} = \mathbf{1} + h\, r_{12} + O(h^2). \tag{3.3.1}$$

Here $\mathbf{1} = I \otimes I$ denotes the $(N^2 \times N^2)$ unit matrix. One says that such $R$-matrices have semiclassical behavior, and $r_{12}$ is called a classical $r$-matrix. It is readily found from the quantum Yang–Baxter equation (3.1.11) that $r_{12}$ satisfies the so-called classical Yang–Baxter equation

$$[r_{12},\ r_{13} + r_{23}] + [r_{13},\ r_{23}] = 0. \tag{3.3.2}$$

Substituting the expansion (3.3.1) in the $RTT$ relations (3.2.1), we obtain

$$[T_1, T_2] = h[T_1T_2, r_{12}] + O(h^2). \tag{3.3.3}$$

This equation demonstrates the fact that the $RTT$ relations (3.2.1) can be interpreted as a quantization (deformation) of the classical Poisson bracket (Sklyanin bracket [43]):

$$\{T_1, T_2\} = [T_1T_2, r_{12}] \tag{3.3.4}$$

(here the elements $T^i_j$ are commutative coordinates of some Poisson manifold). The classical Yang–Baxter equation (3.3.2) guarantees fulfillment of the Jacobi identity for the bracket (3.3.4). From the requirement of antisymmetry of the Poisson bracket (3.3.4), we obtain

$$\{T_1, T_2\} = [T_1T_2, -r_{21}]. \tag{3.3.5}$$

Thus, the classical $r$-matrix $r^{(-)}_{12} = -r_{21}$ corresponding to the representation $R^{(-)}$ (3.2.18) must also be a solution of Eq. (3.3.2), as is readily shown by making the substitution $3 \leftrightarrow 1$ in (3.3.2). On the other hand, comparing (3.3.4) and (3.3.5), we obtain

$$T_1T_2(r_{12} + r_{21}) = (r_{12} + r_{21})T_1T_2. \tag{3.3.6}$$

Thus,

$$t_{12} = \frac{1}{2}(r_{12} + r_{21}) \tag{3.3.7}$$

is an invariant with respect to the adjoint action of the matrix $T_1T_2$ (it is an ad-invariant). We introduce the new classical $r$-matrix

$$\tilde{r}_{12} = \frac{1}{2}(r_{12} - r_{21}). \tag{3.3.8}$$

Then the Sklyanin bracket can be represented in the manifestly antisymmetric form

$$\{T_1, T_2\} = [T_1T_2, \tilde{r}_{12}], \tag{3.3.9}$$

and the matrix $\tilde{r}$ (3.3.8) satisfies the modified classical Yang–Baxter equation

$$[\tilde{r}_{12},\ \tilde{r}_{13} + \tilde{r}_{23}] + [\tilde{r}_{13},\ \tilde{r}_{23}] = \frac{1}{4}\,[r_{23} + r_{32},\ r_{13} + r_{31}] = [t_{23},\ t_{13}]. \tag{3.3.10}$$

Note that the reflection equation algebras (3.2.31), (3.2.32) can also be regarded as the result of quantization of a certain Poisson structure. For example, for these algebras, after substitution of (3.3.1), we have [121] (see also [88])

$$\{L_2,\ L_1\} = [L_1,\ [L_2,\ \tilde{r}_{12}]] + L_1t_{12}L_2 - L_2t_{12}L_1,$$

$$\{\overline{L}_2,\, \overline{L}_1\} = -[\overline{L}_1,\, [\overline{L}_2,\, \tilde{r}_{12}]] + \overline{L}_1 t_{12} \overline{L}_2 - \overline{L}_2 t_{12} \overline{L}_1,$$

where again we must assume that $[L_1 L_2,\, t_{12}] = 0 = [\overline{L}_1 \overline{L}_2,\, t_{12}]$ (cf. (3.3.6)). On the other hand, the relations (3.2.37) in the zeroth order in $h$ give the equations

$$[\tilde{L}_1,\, \tilde{L}_2] = [t_{12},\, \tilde{L}_1], \quad \Big([t_{12},\, \tilde{L}_1 + \tilde{L}_2] = 0\Big),$$

and this enables us to regard (3.2.37) as a deformation of the defining relations of a Lie algebra.

Now we consider the universal enveloping $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ with defining relations (2.2.4) as a bialgebra (2.2.5) and assume that the cocommutative comultiplication $\Delta$ (2.2.5) is quantized $\Delta \to \Delta_h$ in such a way that $\Delta_h$ is noncocommutative. The semiclassical expansion of $\Delta_h$ is[9]

$$\Delta_h(J_\alpha) = J^1_\alpha + J^2_\alpha + h\, \mu^{\beta\gamma}_\alpha\, J^1_\beta\, J^2_\gamma + h^2\, \left(\mu^{\beta_1\beta_2,\gamma}_\alpha\, J^1_{\beta_1}\, J^1_{\beta_2}\, J^2_\gamma + \mu^{\beta,\gamma_1\gamma_2}_\alpha\, J^1_\beta\, J^2_{\gamma_1}\, J^2_{\gamma_2}\right) + h^3 \ldots \quad (3.3.11)$$

where $J^1_\alpha = J_\alpha \otimes 1$, $J^2_\alpha = 1 \otimes J_\alpha$, the term of zeroth order in $h$ in (3.3.11) is the classical comultiplication (2.2.5) and $\mu^{\beta\gamma}_\alpha, \mu^{\beta_1\beta_2,\gamma}_\alpha, \ldots$ are some constants. The comultiplication map (3.3.11) (as well as the opposite comultiplication $\Delta^{\mathsf{op}}_h$; see (2.2.2)) should be a homomorphic map for the Lie algebra (2.2.4):

$$[\Delta_h(J_\alpha),\, \Delta_h(J_\beta)] = t^\gamma_{\alpha\beta}\, \Delta_h(J_\gamma), \quad [\Delta^{\mathsf{op}}_h(J_\alpha),\, \Delta^{\mathsf{op}}_h(J_\beta)] = t^\gamma_{\alpha\beta}\, \Delta^{\mathsf{op}}_h(J_\gamma). \quad (3.3.12)$$

Then the subtraction of the second relation of (3.3.12) from the first one gives the following equation:

$$[\Delta^-_h(J_\alpha),\, \Delta^+_h(J_\beta)] + [\Delta^+_h(J_\alpha),\, \Delta^-_h(J_\beta)] = t^\gamma_{\alpha\beta}\, \Delta^-_h(J_\gamma)$$

(here we define $\Delta^-_h := \Delta_h - \Delta^{\mathsf{op}}_h$ and $\Delta^+_h := \frac{1}{2}\,(\Delta_h + \Delta^{\mathsf{op}}_h)$) which is rewritten (in the first order of $h$) as

$$[\delta(J_\alpha),\, J^1_\beta + J^2_\beta] + [J^1_\alpha + J^2_\alpha,\, \delta(J_\beta)] = t^\gamma_{\alpha\beta}\, \delta(J_\gamma), \quad (3.3.13)$$

where the map $\delta\colon\, g \to g \wedge g$ is

$$\delta(J_\alpha) = \delta^{\beta\gamma}_\alpha\, J_\beta \otimes J_\gamma, \quad \delta^{\beta\gamma}_\alpha := \mu^{\beta\gamma}_\alpha - \mu^{\gamma\beta}_\alpha. \quad (3.3.14)$$

Equation (3.3.13) is nothing but the cocycle condition for $\delta^{\beta\gamma}_\alpha$:

$$\left(\delta^{\rho\mu}_\alpha\, t^\kappa_{\beta\rho} - \delta^{\rho\kappa}_\alpha\, t^\mu_{\beta\rho}\right) - \left(\delta^{\rho\mu}_\beta\, t^\kappa_{\alpha\rho} - \delta^{\rho\kappa}_\beta\, t^\mu_{\alpha\rho}\right) = t^\gamma_{\alpha\beta}\, \delta^{\mu\kappa}_\gamma.$$

On the other hand, the structure constants $(\Delta^-)^{ij}_k = \Delta^{ij}_k - \Delta^{ji}_k$ satisfy the co-Jacobi identity

$$(\Delta^-)^{jk}_i\, (\Delta^-)^{nm}_j + (\Delta^-)^{jn}_i\, (\Delta^-)^{mk}_j + (\Delta^-)^{jm}_i\, (\Delta^-)^{kn}_j = 0,$$

as it is evident from the coassociativity condition (2.1.8). This identity for the comultiplication (3.3.11) in the order $h^2$ reduces to the co-Jacoby identity for the structure constants $\delta^{\beta\gamma}_\alpha$ (3.3.14):

$$\delta^{\beta\gamma}_\alpha\, \delta^{\rho\xi}_\beta + (\text{cycle}\ \ \gamma, \rho, \xi) = 0. \quad (3.3.15)$$

Thus, we have arrived to the following definition [10].

[9] The terms $h\phi^{\beta\gamma}_\alpha J^1_\beta J^1_\gamma$ and $h\phi^{\beta\gamma}_\alpha J^2_\beta J^2_\gamma$ are gauged out by triviality transformation from this expansion (see, e.g., [68]).

**Definition 9.** *The vector space $\mathfrak{g}$ with the basis $\{J_\alpha\}$ equipped with a linear map $\delta\colon g \to g \wedge g$ (3.3.14) satisfying the co-Jacobi identity (3.3.15) is called a Lie coalgebra. A Lie bialgebra is a Lie algebra (2.2.4) which is at the same time a Lie coalgebra with the map $\delta\colon g \to g \wedge g$ (3.3.14), (3.3.15) satisfying the cocycle condition (3.3.13).*

Let $\mathfrak{g}$ be a Lie bialgebra. If there exists an element $r \in g \otimes g$ such that the map $\delta$ has the form

$$\delta(J) = [J \otimes 1 + 1 \otimes J,\, r] \quad \forall J \in g,$$

then the Lie bialgebra $\mathfrak{g}$ is called a coboundary or $r$-matrix bialgebra.

### *3.4. The quantum groups $GL_q(N)$, $SL_q(N)$ and their quantum algebras and hyperplanes*

#### *3.4.1. $GL_q(N)$ quantum hyperplanes and $R$-matrices*

In this subsection, we discuss the simplest nontrivial quantum groups, which are the quantizations (deformations) of the linear Lie groups $GL(N)$ and $SL(N)$. We begin with the definition of a quantum hyperplane. We recall that the Lie group $GL(N)$ is the set of nondegenerate $N \times N$ matrices $T^i_j$ that act on an $N$-dimensional vector space, whose coordinates we denote by $x^i$ $(i = 1, \dots N)$. Thus, we have the transformations

$$x^i \to \tilde{x^i} = T^i_j x^j, \tag{3.4.1}$$

which we can regard from a different point of view. Namely, let $\{T^i_j\}$ and $\{x^i\}$ $(i, j = 1, \dots, N)$ be the generators of two Abelian (commuting) algebras

$$[x^i, x^j] = [T^i_j, T^k_l] = [T^i_j, x^k] = 0. \tag{3.4.2}$$

Then the transformation (3.4.1) can be regarded as an action (more precisely, it is a coaction) of the algebra $\{T\}$ on the algebra $\{x\}$:

$$x^i \to \delta_T(x^i) \equiv \tilde{x^i} = T^i_j \otimes x^j \tag{3.4.3}$$

that preserves the Abelian structure of the latter, i.e., we have $[\tilde{x}^i,\, \tilde{x}^j] = 0$. We introduce a deformed $N$-dimensional "vector space" whose coordinates $\{x^i\}$ commute as follows:

$$x^i x^j = q x^j x^i,\ i < j, \tag{3.4.4}$$

where $q$ is some number (the deformation parameter). In other words, we now have a noncommutative associative algebra with $N$ generators $\{x^i\}$. In accordance with (3.4.4), any element of this algebra, which is a monomial of arbitrary degree

$$x^{i_1} x^{i_2} \cdots x^{i_K}, \tag{3.4.5}$$

can be uniquely ordered lexicographically, i.e., in such a way that $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_K$. Of such algebras, one says that they possess the Poincare–Birkhoff–Witt (PBW) property. An algebra with $N$ generators satisfying (3.4.4) is called an $N$-dimensional quantum hyperplane [73, 74]. The relations (3.4.4) can be written in the matrix form

$$R^{i_1 i_2}_{j_1 j_2} x^{j_1} x^{j_2} = q x^{i_2} x^{i_1} \Leftrightarrow R_{12} x_1 x_2 = q x_2 x_1 \Leftrightarrow \hat{R}\, x_1\, x_2 = q\, x_1\, x_2. \tag{3.4.6}$$

Here the indices 1 and 2 label the vector spaces on which the $R$-matrix, realized in the tensor square $\mathrm{Mat}(N) \otimes \mathrm{Mat}(N) =: \mathrm{Mat}(N)_1 \, \mathrm{Mat}(N)_2$, acts. Thus, the indices 1 and 2 of the $R$-matrix show how the $R$-matrix acts on the direct product of the first and second vector spaces. We emphasize that the $R$-matrix depends on the parameter $q$ and, generally speaking, its explicit form is recovered nonuniquely from the relations (3.4.4). However, if we require that the $R$-matrix (3.4.6) be constructed by means of two $GL(N)$-invariant tensors $\mathbf{1}_{12}$ and $P_{12}$, i.e.,[10]

$$R^{i_1 i_2}_{j_1 j_2} = (\delta^{i_1}_{j_1} \delta^{i_2}_{j_2}) \cdot a_{i_1 i_2} + (\delta^{i_1}_{j_2} \delta^{i_2}_{j_1}) \cdot b_{i_1 i_2}, \tag{3.4.7}$$

and also satisfy the Yang–Baxter equation (3.1.2) and have lower-triangular block form ($R^{i_1 i_2}_{j_1 j_2} = 0$, $i_1 < j_1$), then we obtain the explicit expression [42, 113]

$$R_{12} = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \lambda \sum_{i > j} e_{ij} \otimes e_{ji}, \tag{3.4.8}$$

$$\hat{R}_{12} = P_{12} \, R_{12} = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} + \lambda \sum_{i > j} e_{jj} \otimes e_{ii}, \tag{3.4.9}$$

where $e_{ij}|_{i,j=1,\ldots,N}$ are matrix units: $(e_{ij})^k_l = \delta^{ik} \delta_{jl}$ , $P_{12} := \sum_{k,\ell} e_{k\ell} \otimes e_{\ell k}$ is a permutation matrix and here and below we often use notation $\boxed{\lambda := q - q^{-1}}$ . Equation (3.4.8) is represented in the components in the form

$$\begin{aligned} R^{i_1 i_2}_{j_1 j_2} &= \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} (1 + (q-1) \delta^{i_1 i_2}) + \lambda \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \Theta_{i_1 i_2}, \\ \hat{R}^{i_1 i_2}_{j_1 j_2} &= \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \, q^{\delta_{i_1 i_2}} + \lambda \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \Theta_{i_2 i_1}, \\ \Theta_{ij} &= \{1 \ \text{if} \ \ i > j, \ \ 0 \ \text{if} \ \ i \leqslant j\}. \end{aligned} \tag{3.4.10}$$

It can be verified (by using, e.g., the diagrammatic technique of Subsection **3.6**) that this $R$-matrix satisfies the Hecke relation (3.1.68) (a special case of (3.1.64)):

$$R_{12} - \lambda \, P_{12} - R^{-1}_{21} = 0 \ \Rightarrow \ \hat{R} - \lambda I - \hat{R}^{-1} = 0 \quad \Leftrightarrow \quad \hat{R}^2 = \lambda \hat{R} + I = 0, \tag{3.4.11}$$

where $I^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}$ is a unit operator. The following helpful relations also follow from the explicit form (3.4.8), (3.4.10) for the $GL_q(N)$ $R$-matrix:

$$R_{12}[\frac{1}{q}] = R^{-1}_{12}[q] \quad \Leftrightarrow \quad \hat{R}_{12}[\frac{1}{q}] = \hat{R}^{-1}_{21}[q], \tag{3.4.12}$$

$$R^{t_1 t_2}_{12} = R_{21}, \quad R^{t_1}_{12} \, R_{12} = R_{12} \, R^{t_1}_{12}. \tag{3.4.13}$$

The $R$-matrix (3.4.7) (where without loss of generality one can fix $b_{ii} = 0$) is skew-invertible iff $a_{ij} \neq 0$ ($\forall i, j$) and $\det(||b_{ij} + a_{ii} \delta_{ij}||) \neq 0$. Then the skew-inverse matrix $\Psi_{12}$ (3.1.18) is represented in the form

$$\hat{\Psi}^{i_1 i_2}_{j_1 j_2} = \frac{1}{a_{i_2 i_1}} \, \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} - d_{i_2 i_1} \, \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}, \tag{3.4.14}$$

where coefficients $d_{ij}$ are defined by the matrix equation

$$d = A^{-1} \, B \, (A + B)^{-1}, \quad A_{ij} := a_{ii} \delta_{ij}, \quad B = ||b_{ij}||.$$

[10] The form of $R$-matrix (3.4.7) proves to be very fruitful for the construction of solutions for dynamical Yang–Baxter equations; see [109, 110] and references therein (see also Subsection **3.8** below).

For the given $R$-matrix (3.4.10), the matrix $\hat{\Psi}_{12}$ (3.4.14) is calculated in the form [68]

$$\begin{gathered}\hat{\Psi}^{i_1 i_2}_{j_1 j_2} = q^{-\delta_{i_1 i_2}} \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} - \lambda \Theta_{i_2 i_1} q^{2(i_1 - i_2)} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}, \\ \hat{\Psi}_{12} = q^{-1} \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} - \lambda \sum_{i<j} q^{2(i-j)} e_{ii} \otimes e_{jj}.\end{gathered} \tag{3.4.15}$$

Then the quantum trace matrices $D$, $Q$ (3.1.20) and the related quantum traces (3.2.12) are

$$\begin{gathered}D_1 \equiv \mathrm{Tr}_2 \left( \hat{\Psi}_{12} \right) = \mathrm{diag}\{q^{-2N+1}, q^{-2N+3}, \dots, q^{-1}\}, \quad D^i_j = q^{2(i-N)-1} \delta^i_j, \\ Q_2 \equiv \mathrm{Tr}_1 \left( \hat{\Psi}_{12} \right) = \mathrm{diag}\{q^{-1}, \dots, q^{-2N+3}, q^{-2N+1}\}, \quad Q^i_j = q^{1-2i} \delta^i_j,\end{gathered} \tag{3.4.16}$$

$$\mathrm{Tr}_D A := \mathrm{Tr}(DA) \equiv \sum_{i=1}^{N} q^{2(i-N)-1} A^i_i, \;\; \mathrm{Tr}_Q A := \mathrm{Tr}(QA) \equiv \sum_{i=1}^{N} q^{1-2i} A^i_i.$$

We also note the useful relations (cf. (3.1.22), (3.2.17))

$$\begin{gathered}\mathrm{Tr}_D(I) = \mathrm{Tr}(D) = q^{-N} [N]_q = \mathrm{Tr}(Q) = \mathrm{Tr}_Q(I), \\ q^N \, \mathrm{Tr}_{D(3)} \hat{R}^{\pm 1}_{23} = q^{\pm N} \cdot I_{(2)} = q^N \, \mathrm{Tr}_{Q(1)} \hat{R}^{\pm 1},\end{gathered} \tag{3.4.17}$$

where $[N]_q = \frac{q^N - q^{-N}}{q - q^{-1}}$. One can readily prove the cyclic property of the quantum traces

$$\mathrm{Tr}_{D(12)}(\hat{R} E_{12}) = \mathrm{Tr}_{12}(D_1 D_2 \hat{R} E_{12}) = \mathrm{Tr}_{12}(\hat{R} D_1 D_2 \, E_{12}) = \mathrm{Tr}_{D(12)}(E_{12} \hat{R}), \tag{3.4.18}$$

where $E_{12} \in \mathrm{Mat}(N) \otimes \mathrm{Mat}(N)$ is a matrix with noncommutative entries. In (3.4.18), we have used the fact that the matrix $D$, by definition, obeys Eq. $[\hat{R}, \, D_1 D_2] = 0$ (3.1.36) (note that for $R$-matrices of the type (3.4.7) all diagonal matrices $D$ satisfy this equation). The same cyclic property $\mathrm{Tr}_{Q(12)}(\hat{R} E_{12}) = \mathrm{Tr}_{Q(12)}(E_{12} \hat{R})$ is also valid for the traces $\mathrm{Tr}_Q$.

In semiclassical limit (3.3.1), relation (3.4.11) can be written in the form

$$r_{12} + r_{21} = 2 P_{12}. \tag{3.4.19}$$

Thus, for the Lie–Poisson structure on the group $GL(N)$ the transposition matrix $P_{12}$ is taken as the ad-invariant tensor $t_{12}$. For the $\tilde{r}$-matrix (3.3.8) determining the Sklyanin bracket, we obtain from (3.4.10) the expression

$$\tilde{r}_{12} = \sum_{i>j} [e_{ij} \otimes e_{ji} - e_{ji} \otimes e_{ij}] \in gl(N) \wedge gl(N). \tag{3.4.20}$$

In accordance with (3.1.68), (3.1.66), and (3.1.67), for $q^2 \neq -1$ the matrix $\hat{R}$ has the spectral decomposition

$$\hat{R} = q \mathbf{P}^+ - q^{-1} \mathbf{P}^-, \tag{3.4.21}$$

with projectors

$$\mathbf{P}^{\pm} = (q + q^{-1})^{-1} \{ q^{\mp 1} \mathbf{1} \pm \hat{R} \}, \tag{3.4.22}$$

which are the quantum analogs of the symmetrizer ($\mathbf{P}^+$) and antisymmetrizer ($\mathbf{P}^-$), as can be seen by setting $q = 1$ in (3.4.22). Using the projector $\mathbf{P}^-$, we can represent the definition (3.4.4) of the quantum hyperplane in the form

$$\mathbf{P}^- \, x_1 \, x_2 = 0. \tag{3.4.23}$$

Note that the relations

$$\mathbf{P}^+ \, x_1 \, x_2 = 0 \Leftrightarrow (x^i)^2 = 0, \;\; x^i x^j = -q^{-1} x^j x^i \;\; (i < j) \tag{3.4.24}$$

define a fermionic $N$-dimensional quantum hyperplane that is a deformation of the algebra of $N$ fermions: $x^i x^j = -x^j x^i$.

*3.4.2. Quantum groups* $\mathrm{Fun}(GL_q(N))$, $\mathrm{Fun}(SL_q(N))$ *and q-determinants*

A natural question now is about the properties of the $N \times N$ matrix elements $T^i_j$ that determine the transformations (3.4.3) of the quantum bosonic (3.4.4), (3.4.23) and fermionic (3.4.24) hyperplanes. These properties should be such that the transformed coordinates $\tilde{x}^i$ form the same quantum algebras ($q$-hyperplanes) (3.4.23) and (3.4.24). It is readily seen that the elements of the $N \times N$ matrix $T^i_j$ must satisfy the both conditions

$$\mathbf{P}^+ T_1 T_2 \mathbf{P}^- = 0, \qquad \mathbf{P}^- T_1 T_2 \mathbf{P}^+ = 0. \tag{3.4.25}$$

Indeed, we have for bosonic $x^-$ and fermionic $x^+$ hyperplanes (we omit the symbol $\otimes$ in (3.4.3))

$$0 = \mathbf{P}^\pm \tilde{x}^\pm_1 \tilde{x}^\pm_2 = \mathbf{P}^\pm T_1 T_2 x^\pm_1 x^\pm_2 =$$
$$= \mathbf{P}^\pm T_1 T_2 (\mathbf{P}^+ + \mathbf{P}^-) x^\pm_1 x^\pm_2 = \mathbf{P}^\pm T_1 T_2 \mathbf{P}^\mp x^\pm_1 x^\pm_2,$$

and we deduce (3.4.25) (otherwise new quadratic relations on the coordinates $x^\pm$ should be imposed). Equations (3.4.25) are equivalent to the $RTT$ relations (3.2.1) for the elements of the $N \times N$ quantum matrix $||T^i_j||$:

$$\hat{R} T_1 T_2 - T_1 T_2 \hat{R} = (q + q^{-1})\big(\mathbf{P}^+ T_1 T_2 \mathbf{P}^- - \mathbf{P}^- T_1 T_2 \mathbf{P}^+\big) = 0. \tag{3.4.26}$$

We note that one can define the quantum matrix algebra when only one of two relations in (3.4.25) is fulfilled. In this case, the quantum matrix algebras, generated by $T^i_j$, are called *half-quantum* or *Manin* matrix algebras [107, 108].

For the $R$-matrix (3.4.10) the $RTT$ relations (3.2.1) and (3.4.26) can be written in the component form

$$\begin{gathered} T^i_k T^j_k = q\, T^j_k T^i_k, \quad T^k_i T^k_j = q\, T^k_j T^k_i, \quad (i < j,\; k = 1, \dots, N), \\ [T^{i_1}_{j_1}, T^{i_2}_{j_2}] = (q - q^{-1})\, T^{i_1}_{j_2} T^{i_2}_{j_1}, \quad [T^{i_1}_{j_2}, T^{i_2}_{j_1}] = 0, \quad (i_1 < i_2,\; j_1 < j_2). \end{gathered} \tag{3.4.27}$$

The $RTT$ algebra with defining relations (3.4.27) is a bialgebra with the structure mappings $\Delta, \epsilon$ presented in (3.2.3). The simplest special case ($N = 2$) of this algebra is defined by

$$\begin{gathered} T^1_{\;k} T^2_{\;k} = q\, T^2_{\;k} T^1_{\;k}, \quad T^k_{\;1} T^k_{\;2} = q\, T^k_{\;2} T^k_{\;1}, \quad (k = 1, 2), \\ [T^1_{\;1}, T^2_{\;2}] = (q - q^{-1})\, T^1_{\;2} T^2_{\;1}, \quad [T^1_{\;2}, T^2_{\;1}] = 0. \end{gathered} \tag{3.4.28}$$

One can directly check that $\det_q(T) := T^1_{\;1} T^2_{\;2} - q T^1_{\;2} T^2_{\;1} \equiv T^2_{\;2} T^1_{\;1} - q^{-1} T^1_{\;2} T^2_{\;1}$ is a central element for the algebra (3.4.28). This element is called quantum determinant for $(2\times 2)$ quantum matrix $||T^i_j||$, since for $q = 1$ the element $\det_q(T)$ coincides with the usual determinant. Let $\det_q(T)$ be invertible element. Then the inverse matrix $T^{-1}$ is

$$T^{-1} = \begin{pmatrix} T^2_{\;2} & -q^{-1} T^1_{\;2} \\ -q T^2_{\;1} & T^1_{\;1} \end{pmatrix} \frac{1}{\det_q(T)} \qquad \Rightarrow \qquad \det_{q^{-1}}(T^{-1}) = \det^{-1}_q(T).$$

Now we generalize the definition of the quantum determinant for the case of $(N \times N)$ quantum matrices $||T^i_j||$. We introduce the quantum determinant $\det_q(T)$, which is a deformation of the ordinary determinant and also is a central element for the $RTT$ algebra (3.4.27). For this aim we introduce the $q$-deformed antisymmetric tensors $\mathcal{E}_{j_1 j_2 \dots j_N}$ and $\mathcal{E}^{j_1 j_2 \dots j_N}$ ($\forall j_k = 1, \dots, N$) as follows:

$$\sum_{j_1 \dots j_N = 1}^{N} \mathcal{E}_{j_1 j_2 \dots j_N}\, \mathcal{E}^{j_1 j_2 \dots j_N} = \mathcal{E}_{\langle 12 \dots N} \mathcal{E}^{12 \dots N\rangle} = 1,$$

$$\begin{aligned} \mathcal{E}_{\langle 12\ldots N}\mathbf{P}^+_{k,k+1} = \mathcal{E}_{\langle 12\ldots N}(\hat{R}_{k,k+1}+q^{-1}) = 0, \;\; 1 \leqslant k < N, \\ \mathbf{P}^+_{k,k+1}\mathcal{E}^{12\ldots N\rangle} = (\hat{R}_{k,k+1}+q^{-1})\mathcal{E}^{12\ldots N\rangle} = 0, \;\; 1 \leqslant k < N, \end{aligned} \tag{3.4.29}$$

where we have used concise matrix notations. Namely, we denoted by $12\ldots N\rangle$ and $\langle 12\ldots N$ the sets of incoming and outgoing indices, where $1,2,\ldots N$ are numbers of the $N$-dimensional vector spaces $V_N$, and $\mathbf{P}^+_{k,k+1} = I^{\otimes(k-1)}\otimes\mathbf{P}^+\otimes I^{\otimes(N-k-1)}$ are the symmetrizers (3.4.21) acting in the vector spaces $V_N$ labeled by numbers $k$ and $k+1$. Note that, in view of the *RTT* relations (3.2.1), (3.4.27), the tensors $\mathcal{E}_{\langle 12\ldots N}(T_1T_2\cdots T_N)$ and $(T_1T_2\cdots T_N)\mathcal{E}^{12\ldots N\rangle}$ possess the same symmetry[11] (3.4.29) as the tensors $\mathcal{E}_{\langle 12\ldots N}$ and $\mathcal{E}^{12\ldots N\rangle}$, respectively. Supposing that the $\mathcal{E}$-tensors are unique (up to a normalization), one can write

$$\begin{aligned} \det_q(T)\,\mathcal{E}_{j_1j_2\ldots j_N} = \mathcal{E}_{i_1i_2\ldots i_N}T^{i_1}_{j_1}\cdot T^{i_2}_{j_2}\cdots T^{i_N}_{j_N}, \\ \mathcal{E}^{i_1i_2\ldots i_N}\det_q(T) = T^{i_1}_{j_1}\cdot T^{i_2}_{j_2}\cdots T^{i_N}_{j_N}\mathcal{E}^{j_1j_2\ldots j_N}, \end{aligned} \tag{3.4.30}$$

or in concise matrix notations, we have

$$\det_q(T)\,\mathcal{E}_{\langle 12\ldots N} = \mathcal{E}_{\langle 12\ldots N}\,T_1\cdot T_2\cdots T_N, \qquad \mathcal{E}^{12\ldots N\rangle}\det_q(T) = T_1\cdot T_2\cdots T_N\,\mathcal{E}^{12\ldots N\rangle}, \tag{3.4.31}$$

where $T_m := I^{\otimes(m-1)}\otimes T\otimes I^{\otimes(N-m)}$. The scalar coefficient $\det_q(T)$:

$$\det_q(T) = \mathcal{E}_{\langle 12\ldots N}\,(T_1T_2\cdots T_N)\,\mathcal{E}^{12\ldots N\rangle} = \mathrm{Tr}_{12\ldots N}(A_{1\to N}\,T_1T_2\cdots T_N), \tag{3.4.32}$$

is called the quantum determinant for the $(N\times N)$ quantum matrix $||T^i_j||$. In (3.4.32), we introduced the rank-1 projector

$$\begin{aligned} A_{1\to N} := \mathcal{E}^{12\ldots N\rangle}\mathcal{E}_{\langle 12\ldots N}, \quad A_{1\to N}\,A_{1\to N} = A_{1\to N}, \\ A_{1\to N}\,\mathbf{P}^+_{k,k+1} = \mathbf{P}^+_{k,k+1}\,A_{1\to N} = 0, \;\; 1 \leqslant k < N, \end{aligned} \tag{3.4.33}$$

which acts as a $q$-antisymmetrizer in the tensor product $V_N^{\otimes N}$ of $N$ copies of vector spaces $V_N$. It is worth noting that the $q$-antisymmetrizers $A_{1\to 2} := \mathbf{P}^-_{1,2}$ (3.4.22) and $A_{1\to N}$ are two special representatives of the set of antisymmetrizers $\{A_{1\to m}\}$ $(m = 2,3,\ldots,N)$ which act in the tensor product of $m$ vector spaces $V_N$ and satisfy

$$\begin{aligned} A_{1\to m}\,A_{1\to m} = A_{1\to m}, \\ A_{1\to m}\,\mathbf{P}^+_{k,k+1} = \mathbf{P}^+_{k,k+1}\,A_{1\to m} = 0, \;\; 1 \leqslant k < m. \end{aligned} \tag{3.4.34}$$

All of them can be explicitly constructed in terms of the $R$-matrices (3.4.8), (3.4.10) (see, e.g., [111, 113] and Subsection **3.5** below).

The fact that $\det_q(T)$ is indeed a central element in the *RTT* algebra (3.4.27) can be obtained as follows:

$$\mathcal{E}_{\langle 12\ldots N}\det_q(T)\,T_{N+1} = \mathcal{E}_{\langle 12\ldots N}\,T_1\,T_2\cdots T_N\,T_{N+1} = \tag{3.4.35}$$

$$= \mathcal{E}_{\langle 12\ldots N}\,(R_{1,N+1}\cdots R_{N,N+1})^{-1}T_{N+1}\,T_1\,T_2\cdots T_N\,(R_{1,N+1}\cdots R_{N,N+1}) =$$

$$= q^{-1}\,T_{N+1}\mathcal{E}_{\langle 12\ldots N}\,T_1\,T_2\cdots T_N\,(R_{1,N+1}\cdots R_{N,N+1}) = T_{N+1}\det_q(T)\,\mathcal{E}_{\langle 12\ldots N},$$

[11] It is not true for the half-matrix algebras (see definition after Eq. (3.4.26)).

where we have used the definition (3.4.31), the $RTT$ relations presented in the form $T_m T_{m+1} = R^{-1}_{m,m+1} T_{m+1} T_m R_{m,m+1}$, and the equations

$$\begin{aligned} q I_{N+1}\, \mathcal{E}_{\langle 12\dots N} &= \mathcal{E}_{\langle 12\dots N}\, R_{1,N+1} \cdot R_{2,N+1} \cdots R_{N,N+1} \\ q^{-1} I_{N+1}\, \mathcal{E}_{\langle 12\dots N} &= \mathcal{E}_{\langle 12\dots N}\, R^{-1}_{N+1,1} \cdot R^{-1}_{N+1,2} \cdots R^{-1}_{N+1,N}. \end{aligned} \tag{3.4.36}$$

In fact, we have only used the first equation in (3.4.36). The second one is needed if we apply the $RTT$ relations in different manner: $T_m T_{m+1} = R_{m+1,m} T_{m+1} T_m R^{-1}_{m+1,m}$.

The relations (3.4.36) are deduced from the expressions (3.4.32) for quantum determinants. Indeed, we have

$$\det_q(R^{(\pm)}_{N+1}) = \mathcal{E}_{\langle 12\dots N}\, R^{(\pm)}_{1,N+1} \cdots R^{(\pm)}_{N,N+1} \mathcal{E}^{12\dots N\rangle} = q^{\pm 1}\, I_{N+1}, \tag{3.4.37}$$

where matrices $R^{(\pm)}$ are representations for elements $T^i_j$ which were defined in (3.2.18), (3.2.19). The last equality in (3.4.37) follows from the fact that $R^{(+)}$ and $R^{(-)}$ are, respectively, upper and lower triangular block matrices with diagonal blocks of the form

$$(R^{(\pm)}{}^i_i)^k_l = \delta^k_l q^{\pm \delta_{ik}}.$$

Assume that the quantum determinant (3.4.32) is invertible central element. Consider an extension of the $RTT$ algebra (3.4.27) by the central element $\det_q^{-1}(T)$ which is inverse element for the quantum determinant (3.4.32). Then one can use the $\mathcal{E}$-tensor (3.4.29), the identity $\mathcal{E}_{jj_2\dots j_N}\, \mathcal{E}^{ij_2\dots j_N} = \frac{q^N}{[N]_q}\, D^i_j$ (see Eq. (3.5.7) below; matrix $D$ is defined in (3.4.16)), and the inverse element $\det_q^{-1}(T)$ to find an explicit form for the inverse matrix $T^{-1}$:

$$(T^{-1})^i_j = M^i_k\, (D^{-1})^k_j \det_q^{-1}(T) \quad \Rightarrow \quad T^\ell_i\, (T^{-1})^i_j = \delta^\ell_j, \tag{3.4.38}$$

where $M^{i_1}_{j_1} := q^{-N}[N]_q\, \mathcal{E}_{j_1 j_2 \dots j_N}\, T^{j_2}_{i_2} \cdots T^{j_N}_{i_N}\, \mathcal{E}^{i_1 i_2 \dots i_N}$ are quantum minors of the elements $T^{i_1}_{j_1}$. So, the existence of the inverse matrix $||(T^{-1})^i_j||$ for the $RTT$ algebra with $R$-matrix (3.4.10) is equivalent to the invertibility of the central element $\det_q(T)$. We note that Eq. (3.4.31) can be written as

$$\mathcal{E}_{\langle 12\dots N}\, T_N^{-1} \cdots T_1^{-1} = \det_q^{-1}(T)\, \mathcal{E}_{\langle 12\dots N}\,, \qquad T_N^{-1} \cdots T_1^{-1}\, \mathcal{E}^{12\dots N\rangle} = \mathcal{E}^{12\dots N\rangle} \det_q^{-1}(T). \tag{3.4.39}$$

**Definition 10.** *A Hopf algebra generated by unit element 1, $N^2$ elements $T^i_j$ $(i,j = 1,\dots,N)$ which satisfy relations (3.2.1) with R-matrix (3.4.10) and element $\det_q^{-1}(T)$ is called the algebra of functions on the linear quantum group $GL_q(N)$ and is denoted by $\mathrm{Fun}(GL_q(N))$.*

The structure mappings for the algebra $\mathrm{Fun}(GL_q(N))$ are presented in (3.2.3), where elements $(T^{-1})^i_j$ are defined in (3.4.38).

The algebra $\mathrm{Fun}(SL_q(N))$ can be obtained from the algebra $\mathrm{Fun}(GL_q(N))$ by imposing the additional condition $\det_q(T) = 1$ and, in accordance with (3.4.37), the matrix representations (3.2.19) for $T^i_j \in \mathrm{Fun}(SL_q(N))$ are given by formulas

$$\langle L_2^+, T_1\rangle = \frac{1}{q^{1/N}} R_{12}, \ \ \langle L_2^-, T_1\rangle = q^{1/N} R^{-1}_{21}. \tag{3.4.40}$$

Conversely, formulas (3.2.19), (3.4.40) can be interpreted as matrix representations of elements $(L^\pm)^i_j$ which are generators (see Subsection **3.4.3** below) of the universal enveloping algebras $U_q(g\ell(N))$, $U_q(s\ell(N))$.

**Remark 1.** The complexification of the linear quantum groups can be introduced as follows. We first consider the case of the group $GL_q(N)$ and assume that $q$ is *a real number*. We have to define an involution $*$-operation, or simply $*$-involution (which is the antihomomorphism) on the algebra $\mathrm{Fun}(GL_q(N))$ or, in other words, we must introduce the conjugated algebra $\mathrm{Fun}(\widetilde{GL_q}(N))$ with generators[12]

$$\tilde{T} = (T^\dagger)^{-1}, \qquad T^\dagger := (T^*)^t \;\;\Leftrightarrow\;\; (T^\dagger)^i_j := (T^j_i)^*, \tag{3.4.41}$$

and defining relations identical to (3.2.1):

$$R_{12}\,\tilde{T}_1\,\tilde{T}_2 = \tilde{T}_2\,\tilde{T}_1\,R_{12} \quad \Rightarrow \quad \hat{R}_{12}\,\tilde{T}_1\,\tilde{T}_2 = \tilde{T}_1\,\tilde{T}_2\,\hat{R}_{12}. \tag{3.4.42}$$

Then we introduce the extended algebra with generators $\{T^i_j,\ \tilde{T}^k_l\}$ that is the cross (smash) product of the algebras (3.2.1) and (3.4.42) with additional cross-commutation relations (see, for example, [24–26] and [42])

$$\hat{R}\,T_1\,\tilde{T}_2 = \tilde{T}_1\,T_2\,\hat{R}. \tag{3.4.43}$$

It is natural to relate this extended double algebra to $\mathrm{Fun}(GL_q(N,\mathbb{C}))$.

The case of $SL_q(N,\mathbb{C})$ can be obtained from $GL_q(N,\mathbb{C})$ by imposing two subsidiary conditions on the central elements:

$$\det_q(T) = 1, \qquad \det_q(\tilde{T}) = 1. \tag{3.4.44}$$

The real form $U_q(N)$ is extracted from $GL_q(N,\mathbb{C})$ if we require

$$T = \tilde{T} = (T^\dagger)^{-1} \tag{3.4.45}$$

and if, in addition to this, we impose the conditions (3.4.44), then the group $SU_q(N)$ is distinguished.

In the case $|q| = 1$, the definition of $*$-involutions on the linear quantum groups $GL_q(N)$ and $SL_q(N)$ is a nontrivial problem that can be solved only after an imbedding of these quantum groups into the algebra of functions on their cotangent bundles (see Remark 2 in the next subsection).

*3.4.3. Quantum algebras $U_q(gl(N))$ and $U_q(sl(N))$. Universal $\mathcal{R}$-matrix for $U_q(\mathfrak{g})$*

The quantum universal enveloping algebras $U_q(gl(N))$ and $U_q(sl(N))$ appear in the $R$-matrix approach [42] as the algebras with defining relations (3.2.21), (3.2.22). To show this, we consider the upper and lower triangular matrices $L^+$, $L^-$ in the form (cf. [42, 112])

$$L^+ = \begin{pmatrix} q^{H_1} & 0 & \dots & 0 \\ 0 & q^{H_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q^{H_N} \end{pmatrix} \begin{pmatrix} 1 & \lambda f_1 & \lambda f_{13} & \dots & * \\ 0 & 1 & \lambda f_2 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \lambda f_{N-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{3.4.46}$$

$$L^- = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\lambda e_1 & 1 & 0 & \dots & 0 \\ -\lambda e_{31} & -\lambda e_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & \dots & \dots & -\lambda e_{N-1} & 1 \end{pmatrix} \begin{pmatrix} q^{-\tilde{H}_1} & 0 & \dots & 0 \\ 0 & q^{-\tilde{H}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q^{-\tilde{H}_N} \end{pmatrix}, \tag{3.4.47}$$

[12] We recall that $(T^{-1})^t \neq (T^t)^{-1}$ in the case of the quantum matrices (see (3.2.7)).

where $e_\alpha$ and $f_\alpha$ denote, respectively, positive and negative root generators of $U_q(sl(N))$. Here we took into account definitions (3.2.20) of matrices $L^\pm$ and the convention that the universal $R$-matrix has the form $\mathcal{R}^{\mathsf{op}} \sim \sum_\alpha r_\alpha(H_i)(f_\alpha \otimes e_\alpha)$ that is in agreement with the low-triangular expression (3.4.8) of the $GL_q(N)$ $R$-matrix. In particular, from (3.4.46) and (3.4.47), we have

$$(L^+)^i_i = q^{H_i}, \quad (L^-)^i_i = q^{-\tilde{H}_i}, \quad (L^+)^i_{i+1} = \lambda\, q^{H_i}\, f_i, \quad (L^-)^{i+1}_i = -\lambda\, e_i\, q^{-\tilde{H}_i}. \tag{3.4.48}$$

For $R$-matrices (3.4.8), (3.4.10) the relations (3.2.21), (3.2.22) are represented in the component form as

$$(L^\pm)^i_k\,(L^\pm)^j_k = q\,(L^\pm)^j_k\,(L^\pm)^i_k, \quad (L^\pm)^k_i\,(L^\pm)^k_j = q\,(L^\pm)^k_j\,(L^\pm)^k_i, \;\; (i>j), \tag{3.4.49}$$

$$[(L^\pm)^{i_1}_{j_1},\,(L^\pm)^{i_2}_{j_2}] = \lambda\,(L^\pm)^{i_1}_{j_2}\,(L^\pm)^{i_2}_{j_1}, \;\; [(L^\pm)^{i_1}_{j_2},\,(L^\pm)^{i_2}_{j_1}] = 0, \;\; (i_1>i_2,\, j_1>j_2), \tag{3.4.50}$$

$$(L^+)^i_k\,(L^-)^j_k = q\,(L^-)^j_k\,(L^+)^i_k, \quad (L^-)^k_i\,(L^+)^k_j = q\,(L^+)^k_j\,(L^-)^k_i, \;\; (i<j), \tag{3.4.51}$$

$$[(L^\mp)^{i_1}_{j_1},\,(L^\pm)^{i_2}_{j_2}] = 0, \;\; (i_1>i_2,\, j_1>j_2), \quad [(L^+)^i_i,\,(L^-)^i_i] = 0, \tag{3.4.52}$$

$$[(L^-)^{i_1}_{j_1},\,(L^+)^{i_2}_{j_2}] = \lambda\,\left((L^+)^{i_1}_{j_2}\,(L^-)^{i_2}_{j_1} - (L^-)^{i_1}_{j_2}\,(L^+)^{i_2}_{j_1}\right), \;\; (i_1>i_2,\, j_1<j_2) \tag{3.4.53}$$

(there is no summation over repeated indices). We have written only the terms and relations which survive under the condition that $(L^+)^i_{\;j} = 0 = (L^-)^j_{\;i}$, $i>j$.

The substitution of (3.4.48) into Eqs. (3.4.49)–(3.4.53) gives the Drinfeld–Jimbo [113] formulation of $U_q(gl(N))$. Indeed, from Eqs. (3.4.49), (3.4.51), and (3.4.52) one can obtain that $q^{H_i-\tilde{H}_i}$ are the central elements. Thus, the matrices $L^\pm$ can be renormalized (by multiplying them with diagonal matrices) in such a way that elements $q^{H_i-\tilde{H}_i}$ are fixed as units, i.e., $H_i = \tilde{H}_i$. Then from Eq. (3.4.51) we find

$$f_i q^{H_j} = q^{\delta_{j,i}-\delta_{j,i+1}}\, q^{H_j}\, f_i, \quad e_i q^{H_j} = q^{\delta_{j,i+1}-\delta_{j,i}}\, q^{H_j} e_i. \tag{3.4.54}$$

The first equation in (3.4.52) gives $e_i f_j = f_j e_i$ for $i \neq j$ and, taking into account (3.4.53), we derive

$$e_i f_j - f_j e_i = \delta_{i,j}\,\frac{q^{H_i-H_{i+1}} - q^{H_{i+1}-H_i}}{\lambda}. \tag{3.4.55}$$

The first equation in (3.4.50) yields a part of Serre relations

$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i, \;\; (|i-j| \geqslant 2) \tag{3.4.56}$$

and gives the expressions of the composite roots via the simple roots $\{e_i, f_j\}$:

$$\begin{aligned} f_{i-1,i+1} &= (f_i f_{i-1} - q^{-1} f_{i-1} f_i) = \lambda^{-1}\, q^{-H_{i-1}}\,(L^+)^{i-1}_{i+1}, \\ e_{i+1,i-1} &= (e_{i-1} e_i - q e_i e_{i-1}) = -\lambda^{-1}\,(L^-)^{i+1}_{i-1}\, q^{H_{i-1}}. \end{aligned} \tag{3.4.57}$$

Using these definitions and Eqs. (3.4.49), we deduce another part of Serre relations

$$\begin{aligned} &e_i^2 e_{i\pm 1} - (q+q^{-1}) e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0 \;\; (1 \leqslant i, i\pm 1 \leqslant N), \\ &f_i^2 f_{i\pm 1} - (q+q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0 \;\; (1 \leqslant i, i\pm 1 \leqslant N). \end{aligned} \tag{3.4.58}$$

So, we see that Eqs. (3.2.21) and (3.2.22) (with the form of $L^\pm$ given in (3.4.46), (3.4.47)) not only yield the commutation relations (3.4.54), (3.4.55) for the elements of the Chevalley basis, but also present Serre relations (3.4.56), (3.4.58) and define the composite root elements (3.4.57) as the $q$-commutators of the simple root elements. In this sense, the generators $(L^{\pm})^i_j$ (3.4.46), (3.4.47) play the role of a quantum analog of elements of the Cartan–Weyl basis for $U_q(gl(N))$, where $q^{H_k}$, $(L^+)^i_j$ and $(L^-)^j_i$ $(i < j)$ are, respectively, analogs of Cartan elements, negative and positive root generators. The quantum Casimir operators are given by Eqs. (3.2.28) and (3.2.29). The comultiplication, antipode and coidentity in terms of the generators $\{H_i, e_i, f_i\}$ can be deduced from (3.2.23), (3.2.24)

$$\Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes q^{H_{i+1}-H_i}, \quad \Delta(e_i) = e_i \otimes 1 + q^{H_i - H_{i+1}} \otimes e_i,$$
$$S(H_i) = -H_i, \quad S(e_i) = -q^{H_{i+1}-H_i} e_i, \quad S(f_i) = -f_i q^{H_i - H_{i+1}}, \quad \varepsilon(H_i, e_i, f_i) = 0.$$

Note that $\sum_i H_i$ is a central element in the algebra $U_q(gl(N))$ and the condition $\sum_i H_i = 0$ reduces $U_q(gl(N))$ to the algebra $U_q(sl(N))$ with generators $\{h_i := H_i - H_{i+1},\ e_i,\ f_i\}$ subject to the relations

$$[q^{h_i}, q^{h_j}] = 0, \quad q^{h_j} f_i = q^{-a_{ij}} f_i q^{h_j}, \quad q^{h_j} e_i = q^{a_{ij}} e_i q^{h_j}, \tag{3.4.59}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}} \tag{3.4.60}$$

and Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q^{d_i}} (e_i)^k e_j (e_i)^{1-a_{ij}-k} = 0, \quad (e_i \to f_i), \tag{3.4.61}$$

where

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \;\; [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \;\; [k]_q! := [1]_q [2]_q \cdots [k]_q, \;\; [0]_q! := 1, \tag{3.4.62}$$

$a_{ij} = 2\delta_{ij} - \delta_{ji+1} - \delta_{ij+1}$ is Cartan matrix for $sl(N)$, $d_i$ are smallest positive integers (from the set 1,2,3) such that $d_i a_{ij} \equiv a^{\mathsf{sym}}_{ij}$ is symmetric Cartan matrix (for $sl(N)$ case $d_i = 1$). For the quantum algebra (3.4.59)–(3.4.61) the structure mappings are

$$\begin{gathered} \Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i}, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes q^{-d_i h_i}, \quad \Delta(e_i) = e_i \otimes 1 + q^{d_i h_i} \otimes e_i, \\ S(h_i) = -h_i, \quad S(e_i) = -q^{-d_i h_i} e_i, \quad S(f_i) = -f_i q^{d_i h_i}, \quad \varepsilon(h_i, e_i, f_i) = 0. \end{gathered} \tag{3.4.63}$$

**Remark 1.** By making use of the statements of Proposition **3.6**, we construct the central elements (3.2.29) for the algebras $U_q(sl(N))$ as

$$C_{(M)} = \mathrm{Tr}_D \left( L^M \right) \equiv \mathrm{Tr}_Q \left( \bar{L}^M \right), \tag{3.4.64}$$

where $L = S(L^-)L^+$, $\bar{L} = L^+ S(L^-)$ and the quantum trace matrices $D$ and $Q$ are defined in (3.4.16). The elements (3.4.64) are quantum analogs of the Casimir operators for the algebras $U_q(sl(N))$.

**Remark 2.** The relations (3.4.59)–(3.4.61) are used for the Drinfeld–Jimbo [10, 114] formulation of the quantum universal enveloping algebra $U_q(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$ (the elements $e_i, f_i, q^{h_i}$ are related to the simple root $\alpha_i|_{i=1,\dots,r}$ of the Lie algebra $\mathfrak{g}$ of the rank $r$).

By using this formulation of the quasitriangular Hopf algebra $U_q(\mathfrak{g})$, one can explicitly construct the corresponding universal $\mathcal{R}$-matrix (the definition via canonical element is given in (2.4.9)). In the case of algebra $U_q(sl_N)$, the explicit multiplicative formula for the universal $\mathcal{R}$-matrix has been invented in [122]. This result was generalized in [123] for the case of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is any semisimple Lie algebra. For the case of quantum Lie superalgebras the universal $\mathcal{R}$-matrix has been found in [163]. Finite-dimensional representations for the quantum simple Lie algebras $U_q(\mathfrak{g})$ (3.4.59)–(3.4.61) were considered, e.g., in [115].

Here we give (without proof) the explicit multiplicative formula for the $U_q(\mathfrak{g})$ universal $\mathcal{R}$-matrix, where $\mathfrak{g}$ is a simple Lie algebra. We give this formula in the form proposed by S. Khoroshkin and V. Tolstoy [163] (see also [123, 165]). For this we need the notion of the normal ordering [163] of the system $\Delta_+$ of positive roots of Lie algebra $\mathfrak{g}$. We say that the system $\Delta_+$ is in the normal ordering $\Delta_+^{(n)}$, if each composite root $\alpha+\beta \in \Delta_+$, where $\alpha, \beta \in \Delta_+$, has to be placed in the ordering between $\alpha$ and $\beta$. It is clear that there is an arbitrariness in such a normal ordering $\Delta_+^{(n)}$ of positive roots.

**Proposition 3.10.** *For any quantized Lie algebra $U_q(\mathfrak{g})$ with defining relations (3.4.59)–(3.4.61) and for any normal ordering $\Delta_+^{(n)}$ of the positive root system $\Delta_+$ of $\mathfrak{g}$, the universal $\mathcal{R}$-matrix such that $\mathcal{R}^{-1}\Delta\mathcal{R} = \Delta^{\mathsf{op}}$, where $\Delta$ is the comultiplication (3.4.63), is given by the formula*

$$\mathcal{R} = K \cdot \overrightarrow{\prod_{\beta\in\Delta_+^{(n)}}} \exp_{q_\beta}\big((q-q^{-1})(e_\beta \otimes f_\beta)\big), \tag{3.4.65}$$

$$K := q^{\sum_{ij} d_{ij} h_i \otimes h_j}, \quad \exp_q(x) := \sum_{n\geqslant 0} x^n/(n)_q!, \quad (n)_q := (q^n-1)/(q-1),$$

*where $q_\beta = q^{(\beta,\beta)}$, $d_{ij}$ is an inverse matrix for the symmetrized Cartan matrix $a_{ij}^{\mathsf{sym}} = d_i a_{ij}$ (see definition of $d_i$ after (3.4.62)) and the ordered product runs over the normal ordering $\Delta_+^{(n)}$ of the positive roots.*

For the case of $U_q(s\ell(2))$ algebra

$$[e,f] = \frac{q^h - q^{-h}}{q-q^{-1}}, \quad q^h\, f = q^{-2}\, f\, q^h, \quad q^h\, e = q^2\, e\, q^h,$$

the formula (3.4.65) is simplified

$$\mathcal{R} = q^{\frac{1}{2}h\otimes h} \cdot \exp_{q^2}\Big((q-q^{-1})(e\otimes f)\Big).$$

Finally, we note that in the paper [163], the authors used another comultiplication $\Delta'$ for $U_q(\mathfrak{g})$:

$$\Delta'(q^{h_i}) = q^{h_i}\otimes q^{h_i}, \quad \Delta'(f_i) = 1\otimes f_i + f_i \otimes q^{d_i h_i}, \quad \Delta'(e_i) = e_i \otimes 1 + q^{-d_i h_i}\otimes e_i, \tag{3.4.66}$$

which is related to the comultiplication (3.4.63) by twisting $\Delta' = K^{-1}\, P_{12}\, \Delta\, P_{12}\, K \equiv K^{-1}\, \Delta^{\mathsf{op}}\, K$. This explains why our formula (3.4.65) differs by twisting from the formula for $\mathcal{R}$ given in [163].

**Remark 3.** The $*$-involution on the algebra $U_q(sl(N))$ (3.4.59)–(3.4.61) for real $q$ is defined if we note that the algebra with generators $T, \widetilde{T}$ (3.4.1), (3.4.42), (3.4.43) coincides with the $L^\pm$ algebra (3.2.21), (3.2.22) after an identification: $L^- = T^{-1}$, $L^+ = \widetilde{T}^{-1}$. Then, according to

(3.4.41), we require $(L^+)^\dagger = (L^-)^{-1}$. In terms of the Chevalley generators (3.4.46), (3.4.47), it means that

$$h_i^* = h_i, \quad f_i^* = q\, q^{-h_i}\, e_i, \quad e_i^* = q^{-1}\, f_i\, q^{-h_i}. \tag{3.4.67}$$

One can directly check that Eqs. (3.4.59)–(3.4.61) respect the antihomomorphism (3.4.67).

In the case $|q| = 1$, the definition of $*$-involutions on the linear quantum groups and algebras is a nontrivial problem that can be solved [124] only after extension of the algebra of functions on the quantum groups to the algebra of functions on their cotangent bundles, i.e., to the algebra which is a Heisenberg double of $\mathrm{Fun}(GL_q(N))$ and $U_q(gl(N))$ with cross-multiplication rules (3.2.55)–(3.2.57).

### *3.5. Hecke-type R-matrices. Related quantum matrix algebras*

The material in this subsection is based in part on the results of papers [47, 111], [232].

#### *3.5.1. Definitions. (Anti)symmetrizers for Hecke-type R-matrices*

**Definition 11.** *Yang–Baxter R-matrices which obey (3.1.6) and the Hecke condition (3.1.68), (3.4.11) are called Hecke-type R-matrices.*

First of all, we note that the $GL_q(N)$ matrices (3.4.8), (3.4.10) are examples of Hecke-type $R$-matrices, since they satisfy the Hecke condition (3.1.68), (3.4.11). We also note that if $\hat{R}[q]$ satisfies the Hecke condition (3.4.11), then $\hat{R}[-q^{-1}]$ and $-\hat{R}[q^{-1}]$ also satisfy (3.4.11). In this subsection, we present some general facts about Hecke-type $R$-matrices and related quantum algebras.

The antisymmetrizers $A_{1\to m}$ (3.4.34) can be explicitly constructed in terms of Hecke-type $R$-matrices by using the following inductive procedure [113] (the same procedure was used in [111]; see also Subsection **4.3.1** below):

$$A_{1\to k} = A_{2\to k} \left( \frac{\hat{R}_1(q^{k-1})}{[k]_q} \right) A_{2\to k} = A_{1\to k-1} \left( \frac{\hat{R}_{k-1}(q^{k-1})}{[k]_q} \right) A_{1\to k-1} = \tag{3.5.1}$$

$$= \frac{1}{[k]_q!}\, A_{1\to k-1}\, \hat{R}_{k-1}\left(q^{k-1}\right)\, \hat{R}_{k-2}\left(q^{k-2}\right) \dots \hat{R}_2\left(q^2\right)\, \hat{R}_1\left(q\right), \quad (k = 2, 3, \dots N),$$

where $A_{1\to 1} = 1$, $\hat{R}(x) = (x^{-1}\hat{R} - x\hat{R}^{-1})/\lambda$ — Baxterized $R$-matrix (see Subsection **3.8** below), $\hat{R}$ is a Hecke-type $R$-matrix, $[k]_q = (q^k - q^{-k})/\lambda$ and as usual

$$\hat{R}_k = I^{\otimes(k-1)} \otimes \hat{R} \otimes I^{\otimes(N-k)} \in \mathrm{Mat}(N)^{\otimes(N+1)}. \tag{3.5.2}$$

**Definition 12.** *We say that the Hecke-type R-matrix is of the height $N$, if $A_{1\to M} = 0\ \forall M > N$ and $\mathsf{rank}(A_{1\to N}) = 1$.*

Note that for the $GL_q(N)$-type $R$-matrix (3.4.8), (3.4.10) the operator $A_{1\to N+1} = 0$, and $A_{1\to N}$ is the highest $q$-antisymmetrizer in the sequence of the antisymmetrizers (3.5.1). Moreover, we have $\mathsf{rank}(A_{1\to N}) = 1$ in this case. The latter can easily be understood by considering the fermionic quantum hyperplane (3.4.24). Since the operators $A_{1\to k}$ (3.5.1) satisfy (cf. (3.4.29))

$$\hat{R}_j\, A_{1\to k} = A_{1\to k}\, \hat{R}_j = -q^{-1}\, A_{1\to k} \ \ (j = 1, \dots, k-1), \tag{3.5.3}$$

they are symmetry operators for the $k$th order monomials $x^{i_1} \dots x^{i_k}$ in the $q$-fermionic algebra (3.4.24). In view of the explicit relations (3.4.24), one can conclude that there is only one independent monomial of the order $N$ and all monomials $x^{i_1} \dots x^{i_k}$, for $k > N$, are equal to zero. This statement is equivalent to the conditions $\mathsf{rank}(A_{1\to N}) = 1$ and $A_{1\to N+1} = 0$.

In view of the definition (3.5.1), the condition $A_{1\to N+1} = 0$ leads to (for the arbitrary Hecke $\hat{R}$-matrix):

$$A_{1\to N}\, \hat{R}_N^{\pm 1}\, A_{1\to N} = \frac{q^{\pm N}}{[N]_q}\, A_{1\to N}\, I_{N+1}, \quad A_{2\to N+1}\, \hat{R}_1^{\pm 1}\, A_{2\to N+1} = \frac{q^{\pm N}}{[N]_q}\, A_{2\to N+1}\, I_1 \, . \tag{3.5.4}$$

In the case of skew-invertible Hecke $\hat{R}$-matrices, by applying (3.1.18) to Eqs. (3.5.4), we obtain

$$A_{1\to N}\, P_{_{N\,N+1}}\, A_{1\to N} = \frac{q^N}{[N]_q}\, A_{1\to N}\, Q_{_{N+1}}, \quad A_{2\to N+1}\, P_{_{1\,2}}\, A_{2\to N+1} = \frac{q^N}{[N]_q} D_1\, A_{2\to N+1}, \tag{3.5.5}$$

and for completely invertible $\hat{R}$-matrices we have in addition $\overline{Q} = q^{2N}\, Q$, $\overline{D} = q^{2N}\, D$. Acting, respectively, by $\mathrm{Tr}_{_{N+1}}$ and $\mathrm{Tr}_1$ to the first and second equation in (3.5.5), we deduce (cf. (3.4.17)):

$$\mathrm{Tr}(Q) = \mathrm{Tr}(D) = q^{-N}[N]_q \;\; \Rightarrow \;\; \mathrm{Tr}(\overline{Q}) = \mathrm{Tr}(\overline{D}) = q^N [N]_q, \tag{3.5.6}$$

while applying $\mathrm{Tr}_{(1\dots N)}$ and $\mathrm{Tr}_{(2\dots N+1)}$ to the same equations, we obtain [47]:

$$\mathrm{Tr}_{(1\dots N-1)} A_{1\to N} = \frac{\mathsf{rank}(A_{1\to N})}{\mathrm{Tr}(Q)} Q_N, \qquad \mathrm{Tr}_{(2\dots N)} A_{1\to N} = \frac{\mathsf{rank}(A_{1\to N})}{\mathrm{Tr}(D)} D_1. \tag{3.5.7}$$

On the other hand, applying quantum traces $\mathrm{Tr}_{D(N-k+1\dots N)}$ and $\mathrm{Tr}_{Q(1\dots k)}$ to the antisymmetrizers $A_{1\dots N}$, we deduce [47] ($0 \leqslant k \leqslant N$):

$$\begin{bmatrix} N \\ k \end{bmatrix}_q \mathrm{Tr}_{D(k+1\dots N)}\, (A_{1\dots N}) = q^{(k-N)\,N}\, A_{1\dots k}, \qquad A_{1\dots k}|_{k=0} \; := 1, \tag{3.5.8}$$

$$\begin{bmatrix} N \\ k \end{bmatrix}_q \mathrm{Tr}_{Q(1\dots k)}\, (A_{1\dots N}) = q^{-k\,N}\, A_{k+1\dots N} \;, \qquad A_{k+1\dots N}|_{k=N} \; := 1, \tag{3.5.9}$$

where $q$-binomial coefficients are defined in (3.4.62) and we have used Eqs. (3.5.1) and identities

$$\mathrm{Tr}_{_{D(k+1)}} \hat{R}_k(x) = \mathrm{Tr}_{_{Q(k-1)}} \hat{R}_{k-1}(x) = \left( \frac{x^{-1} - x}{\lambda} + x\,\mathrm{Tr}(D) \right) I_k = \frac{x^{-1} - x q^{-2\,N}}{\lambda}\, I_k, \tag{3.5.10}$$

which follow from (3.1.22), (3.5.6). In view of Eqs. (3.1.36), matrices $D$ and $Q$ can be considered as one-dimensional representations of the $RTT$ algebra (3.2.1): $\rho_D(T^i_j) = D^i_j$, $\rho_Q(T^i_j) = Q^i_j$. Thus, we have

$$A_{1\dots N} D_1 D_2 \dots D_N = \det_q(D)\, A_{1\dots N}, \quad A_{1\dots N} Q_1 Q_2 \dots Q_N = \det_q(Q)\, A_{1\dots N}, \tag{3.5.11}$$

and taking $k = 0$ in (3.5.8) and $k = N$ in (3.5.9), we obtain

$$\det_q(D) = q^{-N^2}, \quad \det_q(Q) = q^{-N^2}. \tag{3.5.12}$$

For the Hecke-type $R$-matrix one can construct (in addition to the $q$-antisymmetrizer $A_{1\to k}$ (3.5.1)) the $q$-symmetrizer $S_{1\to k}$:

$$S_{1\to k} = S_{2\to k} \left( \frac{\hat{R}_1(q^{1-k})}{[k]_q} \right) S_{2\to k} = S_{1\to k-1} \left( \frac{\hat{R}_{k-1}(q^{1-k})}{[k]_q} \right) S_{1\to k-1} \tag{3.5.13}$$

(see also Section 4 below). Using identities (3.5.1), (3.5.13), and (3.5.10), one can calculate $q$-ranks for the projectors $A_{1\to k}$ and $S_{1\to k}$:

$$\mathsf{rank}_q(A_{1\to k}) := \mathrm{Tr}_{D(1\dots k)} A_{1\to k} = \frac{(q^{k-1}\mathrm{Tr}(D) - [k-1]_q)}{[k]_q}\, \mathrm{Tr}_{D(1\dots k-1)} A_{1\to k-1} = \dots =$$
$$= \frac{1}{[k]_q!} \prod_{m=1}^{k} \left(q^{m-1}\mathrm{Tr}(D) - [m-1]_q\right), \tag{3.5.14}$$

and analogously

$$\mathsf{rank}_q(S_{1\to k}) := \mathrm{Tr}_{D(1\dots k)} S_{1\to k} = \frac{1}{[k]_q!} \prod_{m=1}^{k} \left(q^{1-m}\mathrm{Tr}(D) + [m-1]_q\right). \tag{3.5.15}$$

By substituting (3.5.6) into (3.5.14), (3.5.15), we deduce for the Hecke-type $R$-matrix (of the height $N$) the following "$q$-dimensions" of the antisymmetrizer and symmetrizer:

$$\mathrm{Tr}_{D(1\dots k)} A_{1\to k} = q^{-kN} \begin{bmatrix} N \\ k \end{bmatrix}_q \;\; (k \leqslant N), \quad \mathrm{Tr}_{D(1\dots k)} A_{1\to k} = 0 \;\; (k > N),$$
$$\mathrm{Tr}_{D(1\dots k)} S_{1\to k} = q^{-kN} \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q .$$

The general formula for $q$-dimensions of any Young $q$-symmetrizer (related to any Young diagram), which is rational function of the Hecke-type $R$-matrices[13], is known and can be found in [203, 208, 209] (see Subsection **4.3.6** below).

Sometimes it is convenient to have Eqs. (3.4.36) not only for $GL_q(N)$-type $R$-matrices, but in a more general form which is valid for any Hecke $R$-matrix such that $A_{1\to N+1} = 0$. For this we consider identity (see, e.g., [232]):

$$A_{2\to N+1}\, \hat{R}_1^{\pm 1} \cdot \hat{R}_2^{\pm 1} \cdots \hat{R}_N^{\pm 1} = (-1)^{N-1} q^{\pm 1} [N]_q\, A_{2\to N+1}\, A_{1\to N} \tag{3.5.16}$$

(we demonstrate a connection of (3.4.36) with (3.5.16) below). The mirror counterpart of the relations (3.5.16) is also valid

$$\hat{R}_N^{\pm 1} \cdots \hat{R}_2^{\pm 1} \cdot \hat{R}_1^{\pm 1}\, A_{2\to N+1} = (-1)^{N-1} q^{\pm 1} [N]_q\, A_{1\to N}\, A_{2\to N+1}. \tag{3.5.17}$$

Equations (3.5.16) and (3.5.17) can be readily deduced from the equation

$$A_{2\to N+1}\, \hat{R}_1^{\pm 1} \dots \hat{R}_N^{\pm 1} = \hat{R}_1^{\pm 1} \dots \hat{R}_N^{\pm 1}\, A_{1\to N}, \tag{3.5.18}$$

which is obtained from the fact that antisymmetrizers are expressed in terms of $R$-matrices (3.5.1) and by using identities

$$\hat{R}_k\, (\hat{R}_1^{\pm 1} \dots \hat{R}_N^{\pm 1}) = (\hat{R}_1^{\pm 1} \dots \hat{R}_N^{\pm 1})\, \hat{R}_{k-1}, \qquad \forall\, k = 2, \dots, N,$$

followed from the braid relations (3.1.11), (3.1.15). Acting on (3.5.18) by $A_{1\to N}$ from the left and making use of Eqs. (3.5.3), (3.5.4) and again Eq. (3.5.18), we deduce (3.5.17). Equa-

[13] These symmetrizers are images of the idempotents of the Hecke algebra; see Subsections **4.3.3** and **4.3.4** below.

tion (3.5.16) can be proved in the same way. Multiplying identities (3.5.16) and (3.5.17) by $A_{2\to N}$, respectively, from the right and left, we find

$$A_{1\to N}\, A_{2\to N}\, A_{1\to N} = [N]_q^{-2}\, A_{1\to N}, \quad A_{2\to N}\, A_{1\to N}\, A_{2\to N} = [N]_q^{-2}\, A_{2\to N}, \tag{3.5.19}$$

and then multiplying (3.5.17) by (3.5.16) from the left and (3.5.16) by (3.5.17) from the right, we obtain equations

$$A_{1\to N}(\hat{R}_N \dots \hat{R}_2 \hat{R}_1^2 \hat{R}_2 \dots \hat{R}_N - q^2) = 0, \quad A_{2\to N+1}(\hat{R}_1 \hat{R}_2 \dots \hat{R}_N^2 \dots \hat{R}_2 \hat{R}_1 - q^2) = 0,$$

which followed from (3.5.4) and are equivalent to $A_{1\to N+1} = 0$ (see (4.3.11) below).

The identity (3.5.16) is valid for any Hecke $R$-matrix of the height $N$ and can be transformed into Eq. (3.4.36). Indeed, for the case when $\mathsf{rank}(A_{1\to N}) = 1$ and, thus, $A_{1\to N}$ is given by the first equation in (3.4.33), one can act on (3.5.16) by $\mathcal{E}_{\langle 2\dots N+1}$ from the left and, as a result, the counterpart of (3.4.36) is obtained

$$\mathcal{E}_{\langle 23\dots N+1}\, \hat{R}_1^{\pm 1} \cdot \hat{R}_2^{\pm 1} \cdots \hat{R}_N^{\pm 1} = q^{\pm 1}\, \mathsf{N}_{\langle N+1}^{\;1\rangle}\, \mathcal{E}_{\langle 12\dots N}. \tag{3.5.20}$$

Here we have introduced the matrix

$$\mathsf{N}_{\langle N+1}^{\;1\rangle} := (-1)^{N-1}\, [N]_q\, \mathcal{E}_{\langle 2\dots N+1}\, \mathcal{E}^{1\dots N\rangle}, \tag{3.5.21}$$

which, for the case of $GL_q(N)$ $R$-matrix (3.4.8), is equal to the unit matrix $\mathsf{N}_j^i = \delta_j^i$ (cf. (3.4.36)). Analogously, by acting of $\mathcal{E}^{2\dots N+1\rangle}$ on (3.5.17) from the right, we deduce

$$\hat{R}_N^{\pm 1} \cdots \hat{R}_2^{\pm 1} \cdot \hat{R}_1^{\pm 1}\, \mathcal{E}^{23\dots N+1\rangle} = q^{\pm 1}\, \mathcal{E}^{12\dots N\rangle}\, (\mathsf{N}^{-1})_{\langle 1}^{\;N+1\rangle}, \tag{3.5.22}$$

where matrix

$$(\mathsf{N}^{-1})_{\langle 1}^{\;N+1\rangle} := (-1)^{N-1}\, [N]_q\, \mathcal{E}_{\langle 1\dots N}\, \mathcal{E}^{2\dots N+1\rangle} \tag{3.5.23}$$

is inverse to the matrix (3.5.21) in view of (3.5.19).

*3.5.2. Quantum determinants for RTT and RLRL algebras*

For the $RTT$ algebra defined by the Hecke $R$-matrix of the height $N$ (Definition **12**), one can introduce a generalization of the $GL_q(N)$ $q$-determinant by using the same formulas (3.4.30), (3.4.32):

$$\begin{gathered} \det_q(T)\, \mathcal{E}_{\langle 12\dots N} = \mathcal{E}_{\langle 12\dots N}\, T_1 \cdot T_2 \cdots T_N, \qquad \mathcal{E}^{12\dots N\rangle} \det_q(T) = T_1 \cdot T_2 \cdots T_N\, \mathcal{E}^{12\dots N\rangle}, \\ \det_q(T) = \mathcal{E}_{\langle 12\dots N}\, (T_1 T_2 \cdots T_N)\, \mathcal{E}^{12\dots N\rangle} = \mathrm{Tr}_{12\dots N}(A_{1\to N}\, T_1 T_2 \cdots T_N). \end{gathered} \tag{3.5.24}$$

In the case when matrix $\mathsf{N}_j^i$ is not proportional to the unit matrix, the chain of relations (3.4.35) gives [111, 232]:

$$\begin{aligned} \mathcal{E}_{\langle 1\dots N} \det_q(T)\, T_{N+1} &= \mathcal{E}_{\langle 1\dots N}\, T_1 \dots T_N\, T_{N+1} = \\ &= \mathcal{E}_{\langle 1\dots N}\, \hat{R}_N^{-1} \dots \hat{R}_1^{-1}\, T_1 \dots T_N\, T_{N+1}\, \hat{R}_1 \dots \hat{R}_N = \\ &= q^{-1} (N^{-1})_{\langle 1}^{N+1\rangle}\, T_1\, \mathcal{E}_{\langle 2\dots N+1}\, T_2 \dots T_{N+1}\, \hat{R}_1 \dots \hat{R}_N = \\ &= q^{-1} (\mathsf{N}^{-1})_{\langle 1}^{N+1\rangle}\, T_1 \det_q(T)\, \mathcal{E}_{\langle 2\dots N+1}\, \hat{R}_1 \dots \hat{R}_N = (\mathsf{N}^{-1}\, T \mathsf{N})_{N+1} \det_q(T) \mathcal{E}_{\langle 1\dots N}, \end{aligned} \tag{3.5.25}$$

where an explicit form of $\mathsf{N}^{-1}$ can be extracted from (3.5.23). It means that for $RTT$ algebras defined by general Hecke-type $R$-matrix of the limited height, the element $\det_q(T)$ is not necessary central. However, let the noncentral element $\det_q(T)$ be invertible. In this case, one can also define the inverse matrix $T^{-1}$ (cf. (3.4.38); see also Eq. (6.16) in [232]):

$$(T^{-1})^{1\rangle}_{\langle N+1} = \frac{(-1)^{N-1}}{[N]_q}\, \mathsf{N}_1^{-1} \det{}_q^{-1}(T)\, \mathcal{E}_{\langle 2\dots N+1}\, T_2 \cdots T_N\, \mathcal{E}^{1\dots N\rangle} \;\;\Rightarrow\;\; (T^{-1})^{1\rangle}_{\langle N+1}\, T_{N+1} = I^{1\rangle}_{\langle N+1},$$

where the matrices $\mathsf{N}$ and $\mathsf{N}^{-1}$ are given in (3.5.21) and (3.5.22).

In the case of the Hecke-type $R$-matrix of the height $N$, the structures (3.2.57) and (3.2.61) of the cross-products (doubles) for the $RTT$ and reflection equation algebras

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}, \quad T_1L_2 = \hat{R}_{12}L_1\hat{R}_{12}^{\pm 1}T_1, \quad \hat{R}_{12}L_1\hat{R}_{12}L_2 = L_1\hat{R}_{12}L_1\hat{R}_{12}, \tag{3.5.26}$$

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}, \quad \overline{L}_1\, T_2 = T_2\hat{R}_{12}\overline{L}_2\hat{R}_{12}^{\pm 1}, \quad \hat{R}_{12}\overline{L}_2\, \hat{R}_{12}\overline{L}_2 = \overline{L}_2\hat{R}_{12}\overline{L}_2\hat{R}_{12} \tag{3.5.27}$$

help us [47, 84, 91] to introduce the notion of the quantum determinants $\mathrm{Det}_q(L)$, $\mathrm{Det}_q(\overline{L})$ for the corresponding reflection equation algebras (3.2.31) and (3.2.32) with generators $L$ and $\overline{L}$. It can be done by using the definition (3.5.24) of $\det_q(T)$ for the $RTT$ algebras with the Hecke-type $R$-matrix of the height $N$. In view of the automorphism (3.2.58), the quantum matrix $(L\,T)$ satisfies the same $RTT$ relation (3.2.1) and, thus, one can consider the same quantum determinant $\det_q(.)$ for the quantum matrix $(L\,T)$ as for the matrix $T$. This determinant is divisible from the right by $\det_q(T)$ and the quotient depends on the matrix $L$ only. This quotient is called the quantum determinant for the reflection equation algebra (3.2.31). We consider only the case of the double with structure (3.2.57) with upper signs in (3.5.26), (3.5.27) (the case with lower signs is considered in the same way; see (3.2.63)):

$$\begin{aligned} \mathrm{Det}_q(L) &:= \det{}_q(L\,T)\frac{1}{\det_q(T)} = \mathcal{E}_{\langle 1\dots N}(L_1T_1\, L_2T_2 \dots L_NT_N)\mathcal{E}^{1\dots N\rangle}\, \frac{1}{\det_q(T)} = \\ &= \mathcal{E}_{\langle 1\dots N}(L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}})T_1 \dots T_N\, \mathcal{E}^{1\dots N\rangle}\, \frac{1}{\det_q(T)} = \mathcal{E}_{\langle 1\dots N}(L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}})\, \mathcal{E}^{1\dots N\rangle} = \\ &= \mathcal{E}_{\langle 1\dots N}(L_{\widetilde{N}} \dots L_{\widetilde{2}}\, L_{\widetilde{1}})\mathcal{E}^{1\dots N\rangle} = \mathrm{Tr}_{1\dots N}\left(A_{1\dots N}\, L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}}\right), \end{aligned} \tag{3.5.28}$$

where

$$L_{\widetilde{k+1}} = \hat{R}_k L_{\widetilde{k}} \hat{R}_k, \qquad L_{\widetilde{1}} := L_1 \tag{3.5.29}$$

are operators that form a commutative set $[L_{\widetilde{k}},\, L_{\widetilde{\ell}}] = 0$. The definition (3.5.28) is generalized as follows:

$$L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}}\, A_{1\dots N} = A_{1\dots N}\, \mathrm{Det}_q(L). \tag{3.5.30}$$

For the second type algebra (3.2.32), (3.2.57) (the algebra (3.5.27) with upper sign) the definition is analogous:

$$\begin{aligned} \mathrm{Det}_q(\overline{L}) = \det{}_q^{-1}(T)\det{}_q(T\overline{L}) = \det{}_q^{-1}(T)\, \mathcal{E}_{\langle 1\dots N}T_1 \dots T_N\, (\overline{L}_{\widetilde{1}}\, \overline{L}_{\widetilde{2}} \dots \overline{L}_{\widetilde{N}})\mathcal{E}^{1\dots N\rangle} = \\ = \mathcal{E}_{\langle 1\dots N}(\overline{L}_{\widetilde{N}} \dots \overline{L}_{\widetilde{2}}\, \overline{L}_{\widetilde{1}})\mathcal{E}^{1\dots N\rangle} = \mathrm{Tr}_{1\dots N}\left(A_{1\dots N}\, \overline{L}_{\widetilde{1}}\, \overline{L}_{\widetilde{2}} \dots \overline{L}_{\widetilde{N}}\right), \end{aligned}$$

where $\overline{L}_{\widetilde{k}} = \hat{R}_k\, \overline{L}_{\widetilde{k+1}}\, \hat{R}_k$ , $\overline{L}_{\widetilde{N}} := \overline{L}_N$ and $[\overline{L}_{\widetilde{k}},\, \overline{L}_{\widetilde{\ell}}] = 0$.

Below, we restrict ourselves to considering only the case of the left reflection equation algebra (3.2.31) (and the double (3.5.26) with upper sign), since the case of the right algebra (3.2.32) is investigated analogously. An interesting property of the determinant $\mathrm{Det}_q(L)$ (followed from discrete evolutions (3.2.58), (3.2.59)) is of its multiplicativity [47]:

$$\begin{aligned} \mathrm{Det}'_q(L^{n+1}) &:= \det{}_q(L^{n+1}\,T)\frac{1}{\det_q(T)} = \det{}_q(L^{n+1}\,T)\frac{1}{\det_q(L\,T)}\det{}_q(L\,T)\frac{1}{\det_q(T)} = \\ &= \left(\mathcal{E}_{\langle 1\dots N}(L_{\widetilde{1}})^n\, (L_{\widetilde{2}})^n \dots (L_{\widetilde{N}})^n\, \mathcal{E}^{1\dots N\rangle}\right)\mathrm{Det}_q(L) = \mathrm{Det}'_q(L^n)\,\mathrm{Det}_q(L) = \left(\mathrm{Det}_q(L)\right)^n, \end{aligned} \tag{3.5.31}$$

where, for $n=1$ we have $\mathrm{Det}'_q(L^n)=\mathrm{Det}_q(L)$.

In view of the automorphism (3.2.60) for $n=1$, such that $T\to(L+x)\,T$, one can define (in the same way as in (3.5.28)) a quantum determinant $\mathrm{Det}_q(L;x)$ [84]:

$$\mathrm{Det}_q(L;x):=\mathrm{det}_q((L+x)\,T)\frac{1}{\mathrm{det}_q(T)}=\mathrm{Tr}_{1\dots N}\left(A_{1\dots N}(L_{\widetilde{1}}+x)\dots(L_{\widetilde{N}}+x)\right), \tag{3.5.32}$$

where $x\in\mathbb{C}$ is a parameter and $L_{\widetilde{k}}$ are given in (3.5.29). Thus, we introduce the characteristic polynomial for the quantum matrix $L$. Here we prefer to use the notation $\mathrm{Det}_q(L;x)$ instead of $\mathrm{Det}_q(L+x)$, since the dependence on $(L+x)$ seems to be broken in view of the last expression of (3.5.32). Taking into account (3.4.31), the determinant (3.5.32) can be also derived as follows:

$$\mathcal{E}^{1\dots N\rangle}\,\mathrm{Det}_q(L;x)=(L_1+x)T_1\,(L_2+x)T_2\dots(L_N+x)T_N\,\mathcal{E}^{1\dots N\rangle}\,\frac{1}{\mathrm{det}_q(T)}=$$
$$=\left((L_{\widetilde{1}}+x)\,(L_{\widetilde{2}}+x)\dots(L_{\widetilde{N}}+x)\right)\mathcal{E}^{1\dots N\rangle}. \tag{3.5.33}$$

The expansion of (3.5.32) over the parameter $x$ gives

$$\mathrm{Det}_q(L;x)=\sum_{k=0}^{N}x^k\,a_{N-k}(L). \tag{3.5.34}$$

Here $a_0(L)=1$, $a_N(L)=\mathrm{Det}_q(L)$,

$$a_m(L)=\mathrm{Tr}_{1\dots N}\Bigl(A_{1\dots N}\sum_{1\leqslant k_1<\dots<k_m\leqslant N}L_{\widetilde{k_1}}L_{\widetilde{k_2}}\dots L_{\widetilde{k_m}}\Bigr)=\alpha_N^{(m)}\,\mathrm{Tr}_{1\dots N}\left(A_{1\dots N}L_{\widetilde{1}}\,L_{\widetilde{2}}\cdots L_{\widetilde{m}}\right), \tag{3.5.35}$$

where in the second equality we applied identities (for all $k_1<\dots<k_m$)

$$\mathrm{Tr}_{1\dots N}\Bigl(A_{1\dots N}L_{\widetilde{k_1}}L_{\widetilde{k_2}}\dots L_{\widetilde{k_m}}\Bigr)=q^{-2(k_1+k_2+\dots k_m)+m(m+1)}\mathrm{Tr}_{1\dots N}\left(A_{1\dots N}L_{\widetilde{1}}\,L_{\widetilde{2}}\cdots L_{\widetilde{m}}\right),$$

and [47]

$$\alpha_N^{(m)}=q^{m(m+1)}\sum_{1\leqslant k_1<\dots<k_m\leqslant N}q^{-2(k_1+k_2+\dots k_m)}=q^{m(m-N)}\begin{bmatrix}N\\m\end{bmatrix}_q \tag{3.5.36}$$

($q$-binomial coefficients $\begin{bmatrix}N\\m\end{bmatrix}_q$ were introduced in (3.4.62)). The sums in (3.5.36) are readily calculated by means of their generating function

$$\alpha(t)=\sum_{m=0}^{N}t^{N-m}q^{-m(m+1)}\alpha_N^{(m)}=\prod_{m=1}^{N}(t+q^{-2m}),$$

which leads to the equation $\alpha_{N+1}^{(m)}=\alpha_N^{(m)}+q^{2(m-N-1)}\alpha_N^{(m-1)}$ solved by (3.5.36). We note that by using (3.5.11), (3.5.12) and then evaluating the trace $\mathrm{Tr}_{D(m+1\dots N)}$ by means of (3.5.8), the elements $a_m(L)$ can also be written in the form [47]

$$a_m(L)=\frac{\alpha_N^{(m)}}{\mathrm{det}_q(D)}\,\mathrm{Tr}_{D(1\dots N)}\left(A_{1\dots N}L_{\widetilde{1}}\cdots L_{\widetilde{m}}\right)=q^{m^2}\,\mathrm{Tr}_{D(1\dots m)}\left(A_{1\dots m}L_{\widetilde{1}}\cdots L_{\widetilde{m}}\right). \tag{3.5.37}$$

Then we have

$$L_{\widetilde{1}}L_{\widetilde{2}}\cdots L_{\widetilde{m}} = [L_1(\hat{R}_1L_1)(\hat{R}_2\hat{R}_1L_1)\dots(\hat{R}_{m-1}\dots\hat{R}_1L_1)]\hat{R}_{(1\to m-1)}\dots\hat{R}_{(1\to 2)}\hat{R}_1 = \\ = L_{\underline{1}}L_{\underline{2}}\cdots L_{\underline{m}}(\hat{R}_1\dots\hat{R}_{(m-1\leftarrow 1)})(\hat{R}_{(1\to m-1)}\dots\hat{R}_1) = L_{\underline{1}}L_{\underline{2}}\cdots L_{\underline{m}}\, y_2\, y_3\dots y_m, \tag{3.5.38}$$

and analogously

$$L_{\widetilde{1}}L_{\widetilde{2}}\cdots L_{\widetilde{m}} = L_{\widetilde{m}}\cdots L_{\widetilde{2}}L_{\widetilde{1}} = y_2\, y_3\dots y_m\, L_{\overline{m}}\dots L_{\overline{2}}L_{\overline{1}}, \tag{3.5.39}$$

where we used notation $\hat{R}_{(1\to m)}$ and $\hat{R}_{(m\leftarrow 1)}$ given in (3.2.47), matrices $y_1 = 1$, $y_2 = \hat{R}_1^2, \dots$, $y_{k+1} = \hat{R}_k y_k \hat{R}_k$ define a commutative set $[y_k,\ y_\ell] = 0$ and the elements

$$L_{\underline{k+1}} = \hat{R}_k L_{\underline{k}} \hat{R}_k^{-1}, \qquad L_{\overline{k+1}} = \hat{R}_k^{-1} L_{\overline{k}} \hat{R}_k$$

were introduced in (3.2.42). According to the identities (3.5.38), (3.5.39) and taking into account (3.2.44), one can write (3.5.37) as

$$a_m(L) = q^m\, \mathrm{Tr}_{D(1\dots m)}\, (\, A_{1\to m} L_{\underline{1}} L_{\underline{2}}\cdots L_{\underline{m}}) = q^m\, \mathrm{Tr}_{D(1\dots m)}\, (\, A_{1\to m} L_{\overline{m}}\cdots L_{\overline{2}} L_{\overline{1}})\,. \tag{3.5.40}$$

The elements $a_m(L)$ (3.5.35), (3.5.37), (3.5.40) are central elements for the reflection equation algebra (3.2.31). Indeed, these elements are obtained from the general center elements (3.2.48) by substitution $X = A_{1\dots m}$.

*3.5.3. Differential calculus on the RTT algebras. Quantum group covariant connections and curvatures*

**a. Bicovariant differential algebras and quantum BRST operator**

For the Hecke-type $R$-matrix (3.1.68), (3.4.11) one can define [72–88] (see also references therein) the bicovariant differential complex on the $RTT$ algebra:

$$\hat{R}\, T_1\, T_2 = T_1\, T_2\, \hat{R}, \quad \hat{R}\, dT_1\, T_2 = T_1\, dT_2\, \hat{R}^{-1}, \quad \hat{R}\, dT_1\, dT_2 = -dT_1\, dT_2\, \hat{R}^{-1}, \tag{3.5.41}$$

where $\hat{R} := \hat{R}_{12}$ and generators $dT^i_j$ $(i, j = 1, \dots, N)$ are interpreted as differentials of the elements $T^i_j$. The algebra (3.5.41) is a graded (exterior) Hopf algebra [81] with structure mappings [82, 88]:

$$\Delta(T) = T\otimes T, \quad \epsilon(T) = I, \quad S(T) = T^{-1}, \\ \Delta(dT^i{}_j) = dT^i{}_k \otimes T^k{}_j + T^i{}_k \otimes dT^k{}_j, \quad \epsilon(dT^i{}_j) = 0, \quad S(dT^i{}_j) = -(T^{-1}\, dT\, T^{-1})^i{}_j,$$

where in the first line we use the index-free matrix notation, the grading is $\mathfrak{gr}(T^i_k) = 0$, $\mathfrak{gr}(dT^i_k) = 1$ and $\otimes$ is the graded tensor product. We extend the graded algebra (3.5.41) by adding [79, 80, 84–86] new generators $\partial^i_j := \partial/\partial T^j_i$ (quantum derivatives) and $\mathbf{i}^i{}_j$ (quantum inner derivatives) with commutation relations (cf. (3.5.41)):

$$\hat{R}\, \partial_2\, \partial_1 = \partial_2\, \partial_1\, \hat{R}, \quad \hat{R}\, \mathbf{i}_2\, \partial_1 = \partial_2\, \mathbf{i}_1\, \hat{R}^{-1}, \quad \hat{R}\, \mathbf{i}_2\, \mathbf{i}_1 = -\mathbf{i}_2\, \mathbf{i}_1\, \hat{R}^{-1}\,. \tag{3.5.42}$$

Assume that the matrix $\partial$ is formally invertible. Then the algebra (3.5.42) is a graded Hopf algebra with structure mappings [84]:

$$\Delta(\partial) = \partial\otimes\partial, \quad \epsilon(\partial) = I, \quad S(\partial) = \partial^{-1}, \\ \Delta(\mathbf{i}) = \mathbf{i}\otimes\partial + \partial\otimes\mathbf{i}, \quad \epsilon(\mathbf{i}) = 0, \quad S(\mathbf{i}) = -\partial^{-1}\, \mathbf{i}\, \partial^{-1},$$

where $\mathfrak{gr}(\partial^i_k) = 0$, $\mathfrak{gr}(\mathbf{i}^i_k) = 1$ and $\otimes$ is the graded tensor product. Now we define the cross-product (Heisenberg double) of the algebras (3.5.41) and (3.5.42) with cross-commutation relations

$$\partial_1 \hat{R}\, dT_1 = dT_2\, \hat{R}^{-1}\, \partial_2, \quad T_2\, \hat{R}\, \mathbf{i}_2 = \mathbf{i}_1\, \hat{R}^{-1}\, T_1, \tag{3.5.43}$$

$$\partial_1 \hat{R}^{\bar{\epsilon}}\, T_1 = T_2\, \hat{R}^{-\bar{\epsilon}}\, \partial_2 + I_1 I_2, \quad \mathbf{i}_1\, \hat{R}^{\epsilon}\, dT_1 + dT_2\, \hat{R}^{\epsilon}\, \mathbf{i}_2 = I_1 I_2, \tag{3.5.44}$$

where $\epsilon = \mp 1$ and $\bar{\epsilon} = \pm 1$. These relations are postulated in such a way that they are bicovariant with respect to the left and right coaction of the $RTT$ algebra:

$$T \to T_L \otimes T \otimes T_R, \quad dT \to T_L \otimes dT \otimes T_R, \quad \partial \to T_R^{-1} \otimes \partial \otimes T_L^{-1}, \quad \mathbf{i} \to T_R^{-1} \otimes \mathbf{i} \otimes T_L^{-1}.$$

Here elements of matrices $T_L$ and $T_R$ generate two $RTT$ algebras (the first relation in (3.5.41)). We note that the bicovariance does not fix uniquely the relations (3.5.43), (3.5.44). Actually we have four graded bicovariant algebras $\mathcal{A}^{\epsilon,\bar{\epsilon}}$ with generators $\{T, dT, \partial, \mathbf{i}\}$ and with different choice of signs $\epsilon$ and $\bar{\epsilon}$ in (3.5.44). It was shown in [84, 86], that there are explicit inner automorphisms in $\mathcal{A}^{\epsilon,\bar{\epsilon}}$ which relates all these algebras.

Now we introduce the left $\overline{L}$ and the right $L$ invariant vector fields in the algebra $\mathcal{A}^{\epsilon,\bar{\epsilon}}$, corresponding left $\overline{\Omega} \in \mathcal{A}^{\epsilon,\bar{\epsilon}}$ and the right $\Omega \in \mathcal{A}^{\epsilon,\bar{\epsilon}}$ invariant differential 1-forms, inner derivatives $\overline{\mathcal{I}}$, $\mathcal{I} \in \mathcal{A}^{\epsilon,\bar{\epsilon}}$, and special invariant operators $\overline{W}$, $W \in \mathcal{A}^{\epsilon,\bar{\epsilon}}$:

$$\begin{aligned} \overline{L} &:= I - \bar{\epsilon}\lambda\, \partial\, T, \qquad L := I - \bar{\epsilon}\lambda\, T\, \partial \;\equiv\; T\, \overline{L}\, T^{-1}, \\ \overline{\Omega} &:= T^{-1}\, dT, \quad \overline{\mathcal{I}} := \mathbf{i}\, T, \qquad \Omega := dT\, T^{-1} \;\equiv\; T\, \overline{\Omega}\, T^{-1}, \quad \mathcal{I} := T\, \mathbf{i}, \\ \overline{W} &:= 1 - \epsilon\lambda\, \mathbf{i}\, dT = 1 - \epsilon\lambda\, \overline{\mathcal{I}}\, \overline{\Omega}, \qquad W := 1 - \epsilon\lambda\, dT\, \mathbf{i} = 1 - \epsilon\lambda\, \Omega\, \mathcal{I}, \end{aligned} \tag{3.5.45}$$

where as usual $\lambda = q - q^{-1}$. Last relations in (3.5.41), (3.5.42), and (3.5.43) lead to (see [84]):

$$\overline{W}_1\, dT_2 = dT_2\, \hat{R}^{\epsilon}\, \overline{W}_2\, \hat{R}^{\epsilon}, \quad dT_1\, W_2 = \hat{R}^{\epsilon}\, W_1\, \hat{R}^{\epsilon}\, dT_1, \tag{3.5.46}$$

$$\mathbf{i}_2\, \overline{W}_1 = \hat{R}^{\epsilon}\, \overline{W}_2\, \hat{R}^{\epsilon}\, \mathbf{i}_2, \quad W_2\, \mathbf{i}_1 = \mathbf{i}_1\, \hat{R}^{\epsilon}\, W_1\, \hat{R}^{\epsilon}, \tag{3.5.47}$$

and from these identities we immediately obtain [84]:

$$\overline{W}_1\, W_2 = W_2\, \overline{W}_1, \quad \overline{W}_2\, \hat{R}^{\epsilon}\, \overline{W}_2\, \hat{R}^{\epsilon} = \hat{R}^{\epsilon}\, \overline{W}_2\, \hat{R}^{\epsilon}\, \overline{W}_2, \quad W_1\, \hat{R}^{\epsilon}\, W_1\, \hat{R}^{\epsilon} = \hat{R}^{\epsilon}\, W_1\, \hat{R}^{\epsilon}\, W_1. \tag{3.5.48}$$

Operators (3.5.45) also obey (see, e.g., [79, 84, 86]) the following relations (cf. (3.2.31), (3.2.32), (3.2.57)):

$$\begin{gathered} \left\{ \begin{array}{l} \overline{L}_1\, T_2 = T_2 \hat{R}^{-\bar{\epsilon}} \overline{L}_2 \hat{R}^{-\bar{\epsilon}}, \quad T_1 L_2 = \hat{R}^{\bar{\epsilon}} L_1 \hat{R}^{\bar{\epsilon}} T_1, \\ \partial_2\, \overline{L}_1 = \hat{R}^{-\bar{\epsilon}}\, \overline{L}_2\, \hat{R}^{-\bar{\epsilon}}\, \partial_2, \quad L_2\, \partial_1 = \partial_1\, \hat{R}^{\bar{\epsilon}}\, L_1\, \hat{R}^{\bar{\epsilon}}, \end{array} \right. \Rightarrow \\ \overline{L}_1\, L_2 = L_2\, \overline{L}_1, \quad \hat{R}^{-\bar{\epsilon}}\, \overline{L}_2\, \hat{R}^{-\bar{\epsilon}}\, \overline{L}_2 = \overline{L}_2\, \hat{R}^{-\bar{\epsilon}}\, \overline{L}_2\, \hat{R}^{-\bar{\epsilon}}, \quad \hat{R}^{\bar{\epsilon}}\, L_1\, \hat{R}^{\bar{\epsilon}}\, L_1 = L_1\, \hat{R}^{\bar{\epsilon}}\, L_1\, \hat{R}^{\bar{\epsilon}}, \end{gathered} \tag{3.5.49}$$

and in addition we have

$$\begin{gathered} \left\{ \begin{array}{l} \overline{L}_1\, dT_2 = dT_2 \hat{R}^{-1} \overline{L}_2 \hat{R}, \quad dT_1 L_2 = \hat{R}^{-1} L_1 \hat{R} dT_1, \\ \mathbf{i}_2\, \overline{L}_1 = \hat{R}^{-1}\, \overline{L}_2\, \hat{R}\, \mathbf{i}_2, \quad L_2\, \mathbf{i}_1 = \mathbf{i}_1\, \hat{R}^{-1}\, L_1\, \hat{R}, \end{array} \right. \Rightarrow \\ \hat{R}^{-1}\, \overline{L}_2\, \hat{R}\, \overline{W}_2 = \overline{W}_2\, \hat{R}^{-1}\, \overline{L}_2\, \hat{R}, \quad W_1\, \hat{R}^{-1}\, L_1\, \hat{R} = \hat{R}^{-1}\, L_1\, \hat{R}\, W_1, \\ \hat{R}^{-1}\, \overline{L}_2\, \hat{R}\, \overline{\mathcal{I}}_2 = \overline{\mathcal{I}}_2\, \hat{R}^{-\bar{\epsilon}}\, \overline{L}_2\, \hat{R}^{-\bar{\epsilon}}, \quad W_1\, \hat{R}^{-1}\, L_1\, \hat{R} = \hat{R}^{-1}\, L_1\, \hat{R}\, W_1. \end{gathered} \tag{3.5.50}$$

**Proposition 3.11** *(See [84]). The defining relations (3.5.41), (3.5.42) and (3.5.43), (3.5.44) are in agreement with the formulas*

$$\begin{gathered}\hat{R}T_1(x)T_2(x) = T_1(x)T_2(x)\hat{R}, \quad \hat{R}dT_1(y)T_2(x) = T_1(x)dT_2(y)\hat{R}^{-1}, \\ \hat{R}dT_1(y)\,dT_2(y) = -dT_1(y)\,dT_2(y)\hat{R}^{-1} \;\Leftrightarrow\; \hat{R}^{\epsilon}dT_1(y)\,dT_2(y) = -dT_1(y)\,dT_2(y)\hat{R}^{-\epsilon},\end{gathered} \tag{3.5.51}$$

*where $x, y$ are parameters, $\hat{R} := \hat{R}_{12}$ and*

$$T(x) = \frac{1}{x + T\partial}\,T \equiv \frac{1}{T^{-1}x + \partial}, \qquad dT(y) = \frac{1}{y + dT\,\mathbf{i}}\,dT \equiv dT\,\frac{1}{y + \mathbf{i}\,dT}.$$

**Proof.** We write the first relation in (3.5.51) as

$$(T_2^{-1}\,x + \partial_2)(T_1^{-1}\,x + \partial_1)\,\hat{R} = \hat{R}\,(T_2^{-1}\,x + \partial_2)(T_1^{-1}\,x + \partial_1).$$

The terms of order $x^2$ and $x^0$ give, respectively, first relations in (3.5.41) and (3.5.42). The terms of order $x^1$ yield relation compatible to the first relation in (3.5.44). For the second relation in (3.5.51) we obtain

$$(y + dT_2\mathbf{i}_2)(T_1^{-1}\,x + \partial_1)\,\hat{R}\,dT_1 = dT_2\,\hat{R}^{-1}\,(T_2^{-1}\,x + \partial_2)(y + \mathbf{i}_1 dT_1),$$

and in the orders $(xy)^0$, $x^1$, $(xy)^1$, $y^1$ we, respectively, deduce second relations in (3.5.42), (3.5.43), (3.5.41) and first relation in (3.5.43). Finally, the third relation in (3.5.51) is represented as

$$(y + dT_1\,\mathbf{i}_1)\hat{R}dT_1\frac{1}{(y + \mathbf{i}_1\,dT_1)}\,dT_2 = -dT_1\,\frac{1}{(y + dT_2\mathbf{i}_2)}dT_2\hat{R}^{-1}(y + \mathbf{i}_2 dT_2),$$

and, after the change of parameter $y \to \frac{y-1}{\epsilon\lambda}$, introducing operators $W, \overline{W}$ (3.5.45) and applying Eqs. (3.5.46), we write it in the form

$$(y + \hat{R}^{\epsilon}W_1\hat{R}^{\epsilon})(y + W_1)\hat{R}^{\epsilon}dT_1\,dT_2 = -dT_1\,dT_2\hat{R}^{-\epsilon}(y + \overline{W}_2)(y + \hat{R}^{\epsilon}\overline{W}_2\hat{R}^{\epsilon}).$$

The terms of order $y^2$ give the last relation in (3.5.41). The terms of order $y^0$ and $y^1$ are identities in view of the relations (3.5.46) and (3.5.48) which encode last relations in (3.5.42) and (3.5.44). ∎

**Corollary**. The formulas (3.5.51) (incorporating the whole differential algebra (3.5.41), (3.5.42), and (3.5.43)) have the structure of Eqs. (3.5.41) for which one can easily establish the Poincaré–Birkhoff–Witt (PBW) property. This indicates that the whole differential algebra (3.5.41), (3.5.42) and (3.5.43), (3.5.44) is also of the PBW type (the flat deformation of the differential algebra obtained in the classical limit $q \to 1$, $\hat{R} \to P$). We also note that the signs of powers of $\hat{R}$-matrices are flashing $\hat{R}^{\bar{\epsilon}} \to \hat{R}^{-\bar{\epsilon}}$ in the first relation in (3.5.44) and not flashing $\hat{R}^{\epsilon} \to \hat{R}^{\epsilon}$ in the second relation in (3.5.44). This is important, otherwise relations (3.5.51) are not fulfilled.

Here we present additional commutation relations for the invariant operators (3.5.45). These relations are useful from a technical point of view.

$$\begin{aligned} \overline{\Omega}_1\,T_2 &= T_2\,\hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}^{-1}, & T_1\,\Omega_2 &= \hat{R}_{12}\,\Omega_1\,\hat{R}_{12}\,T_1, \\ \overline{W}_1\,T_2 &= T_2\,\hat{R}\,\overline{W}_2\,\hat{R}^{-1}, & T_1\,W_2 &= \hat{R}\,W_1\hat{R}^{-1}\,T_1, \\ \overline{\mathcal{I}}_1\,T_2 &= \hat{R}\,T_2\,\hat{R}\,\overline{\mathcal{I}}_2, & T_1\,\mathcal{I}_2 &= \hat{R}^{-1}\,\mathcal{I}_1\hat{R}^{-1}\,T_1, \\ \overline{L}_1\,dT_2 &= dT_2\,\hat{R}_{12}^{-1}\,\overline{L}_2\,\hat{R}_{12}, & dT_1\,L_2 &= \hat{R}_{12}^{-1}\,L_1\,\hat{R}_{12}\,dT_1, \\ \hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}^{-1}\,\overline{\Omega}_2 &= -\overline{\Omega}_2\,\hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}, & \hat{R}_{12}\,\Omega_1\,\hat{R}\,\Omega_1 &= -\Omega_1\,\hat{R}\,\Omega_1\,\hat{R}^{-1}, \\ \hat{R}^{-\bar{\epsilon}}\,\overline{L}_2\,\hat{R}^{-\bar{\epsilon}}\,\overline{\Omega}_2 &= \overline{\Omega}_2\,\hat{R}^{-1}\,\overline{L}_2\,\hat{R}, & \Omega_1\,\hat{R}^{\bar{\epsilon}}\,L_1\,\hat{R}^{\bar{\epsilon}} &= \hat{R}^{-1}\,L_1\,\hat{R}\,\Omega_1. \end{aligned} \tag{3.5.52}$$

$$\begin{array}{rl} \hat{R}_{12}\,\overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{\mathcal{I}}_2 = -\overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{\mathcal{I}}_2\,\hat{R}_{12}^{-1}, & \hat{R}_{12}^{-1}\,\mathcal{I}_1\,\hat{R}_{12}^{-1}\,\mathcal{I}_1 = -\mathcal{I}_1\,\hat{R}_{12}^{-1}\,\mathcal{I}_1\,\hat{R}_{12}, \\ \hat{R}_{12}\overline{\mathcal{I}}_2\,\hat{R}_{12}^{\epsilon}\,\overline{\Omega}_2 + \overline{\Omega}_2\,\hat{R}_{12}^{\epsilon}\,\overline{\mathcal{I}}_2\,\hat{R}_{12} = \hat{R}_{12}, & \mathcal{I}_1\,\hat{R}_{12}^{\epsilon}\,\Omega_1\,\hat{R}_{12} + \hat{R}_{12}\,\Omega_1\,\hat{R}_{12}^{\epsilon}\,\mathcal{I}_1 = \hat{R}_{12}, \\ \overline{\Omega}_2\,\hat{R}_{12}^{\epsilon}\,\overline{W}_2\,\hat{R}_{12}^{\epsilon} = \hat{R}_{12}\,\overline{W}_2\,\hat{R}_{12}^{-1}\,\overline{\Omega}_2\,, & \hat{R}_{12}^{\epsilon}\,W_1\,\hat{R}_{12}^{\epsilon}\,\Omega_1 = \Omega_1\,\hat{R}_{12}\,W_1\,\hat{R}_{12}^{-1}, \\ \hat{R}_{12}^{\epsilon}\,\overline{W}_2\,\hat{R}_{12}^{\epsilon}\,\overline{\mathcal{I}}_2 = \overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{W}_2\,\hat{R}_{12}^{-1}\,, & \mathcal{I}_1\,\hat{R}_{12}^{\epsilon}\,W_1\,\hat{R}_{12}^{\epsilon} = \hat{R}_{12}\,W_1\,\hat{R}_{12}^{-1}\,\mathcal{I}_1, \end{array} \tag{3.5.53}$$

where we denote $\overline{\mathcal{I}} = \mathbf{i}\,T$ and $\mathcal{I} = T\,\mathbf{i}$.

Now instead of the left $\overline{L}$ and the right $L$ invariant vector fields, we introduce new left and right invariant operators [84]:

$$\overline{\mathcal{L}} := \overline{W}\,\overline{L} = (1-\epsilon\lambda\,\mathbf{i}\,dT)(1-\bar{\epsilon}\lambda\,\partial\,T), \qquad \mathcal{L} := W\,L = (1-\epsilon\lambda\,dT\,\mathbf{i})(1-\bar{\epsilon}\lambda\,T\,\partial).$$

As we mentioned above, all algebras $\mathcal{A}^{\epsilon,\bar{\epsilon}}$ are equivalent for different choice of the signs $\epsilon, \bar{\epsilon}$. For simplicity, further we consider only the left invariant operators and fix $\epsilon = 1$ and $\bar{\epsilon} = -1$. In accordance with the formulas (3.5.52), (3.5.53) for $\epsilon = 1$ and $\bar{\epsilon} = -1$, we have the following statement.

**Proposition 3.12.** *The complete set of commutation relations for the exterior differential algebra* $\Gamma^{\wedge} \subset \mathcal{A}^{\epsilon,\bar{\epsilon}}|_{\epsilon=-\bar{\epsilon}=1}$ *with generators* $T$, $\overline{\mathcal{L}}$, $\overline{\mathcal{I}}$, $\overline{\Omega}$ *is [79, 84, 90, 91]:*

$$\hat{R}T_1T_2 = T_1T_2\hat{R}, \quad \overline{\Omega}_1\,T_2 = T_2\,\hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}^{-1}, \quad \hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}^{-1}\,\overline{\Omega}_2 = -\overline{\Omega}_2\,\hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}, \tag{3.5.54}$$

$$\begin{array}{c} \overline{\mathcal{L}}_2\hat{R}\overline{\mathcal{L}}_2\hat{R} = \hat{R}\overline{\mathcal{L}}_2\hat{R}\overline{\mathcal{L}}_2, \quad \overline{\Omega}_2\hat{R}\overline{\mathcal{L}}_2\hat{R} = \hat{R}\overline{\mathcal{L}}_2\hat{R}\overline{\Omega}_2, \quad \overline{\mathcal{I}}_2\hat{R}\overline{\mathcal{L}}_2\hat{R} = \hat{R}\overline{\mathcal{L}}_2\hat{R}\overline{\mathcal{I}}_2, \\ \overline{\mathcal{L}}_1T_2 = T_2\hat{R}\overline{\mathcal{L}}_2\hat{R}, \quad \overline{\mathcal{I}}_1\,T_2 = \hat{R}\,T_2\,\hat{R}\,\overline{\mathcal{I}}_2, \quad \hat{R}_{12}\,\overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{\mathcal{I}}_2 = -\overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{\mathcal{I}}_2\,\hat{R}_{12}^{-1}, \end{array} \tag{3.5.55}$$

$$\hat{R}_{12}\overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{\Omega}_2 + \overline{\Omega}_2\,\hat{R}_{12}\,\overline{\mathcal{I}}_2\,\hat{R}_{12} = \hat{R}_{12}, \tag{3.5.56}$$

*where* $\hat{R} := \hat{R}_{12}$ *is the Hecke-type R-matrix.*

We note that, for the Hecke-type $\hat{R}$-matrix, the differential algebra (3.5.54)–(3.5.56) is identical to the differential algebra $\Gamma^{\wedge}$, proposed in the papers [79, 90, 91], up to the relation (3.5.56) which is written in those papers as $\hat{R}_{12}\overline{\mathcal{I}}_2\,\hat{R}_{12}\,\overline{\Omega}_2 + \overline{\Omega}_2\,\hat{R}_{12}\,\overline{\mathcal{I}}_2\,\hat{R}_{12} = -\hat{R}_{12}$. The change of the sign in the right-hand side of the relation (3.5.56) can be achieved by the transformation $\overline{\mathcal{I}} \to -\overline{\mathcal{I}}$.

By using the general construction [89] of the BRST charge for an arbitrary quantum Lie algebra, we have constructed in [90] a BRST operator $\mathsf{Q}$ for the differential algebra (3.5.55) in the following form[14]:

$$\mathsf{Q} = \mathrm{Tr}_{\mathcal{Q}}\left(\overline{\Omega}\,\frac{(\overline{\mathcal{L}}-1)}{\lambda} + \overline{\Omega}\,\overline{\mathcal{L}}\,\frac{\overline{\Omega}\,\overline{\mathcal{I}}}{(1-\lambda\overline{\Omega}\,\overline{\mathcal{I}})}\right) = -\frac{1}{\lambda}\,\mathrm{Tr}_{\mathcal{Q}}(\overline{\Omega}) + \frac{1}{\lambda}\,\mathrm{Tr}_{\mathcal{Q}}\,(\Theta)\,, \tag{3.5.57}$$

where $\Theta := \overline{\Omega}\,\overline{\mathcal{L}}\,(1-\lambda\overline{\Omega}\,\overline{\mathcal{I}})^{-1}$ and $\mathrm{Tr}_{\mathcal{Q}}(X) := q^{2d}\,\mathrm{Tr}(QX)$ is the second quantum trace in (3.1.39). The normalization factor $q^{2d}$ is introduced to have $\mathrm{Tr}_{\mathcal{Q}1}(\hat{R}^{-1}) = I_2$ (see (3.1.28), (3.1.71)). We note that the left invariant operator $\widetilde{W} := (1-\lambda\overline{\Omega}\,\overline{\mathcal{I}})$, appeared in (3.5.57), differs from the operator $\overline{W} = (1-\lambda\overline{\mathcal{I}}\,\overline{\Omega})$ defined in (3.5.45). For the operator $\widetilde{W}$ we have

$$\begin{array}{c} \widetilde{W}_2\,\hat{R}\,\overline{\mathcal{L}}_2\,\hat{R} = \hat{R}\,\overline{\mathcal{L}}_2\,\hat{R}\,\widetilde{W}_2, \quad \widetilde{W}_2\,\hat{R}\,\overline{\Omega}_2\,\hat{R}^{-1} = \hat{R}^{-1}\,\overline{\Omega}_2\,\hat{R}^{-1}\,\widetilde{W}_2, \\ \widetilde{W}_2\,\hat{R}\,\overline{\mathcal{I}}_2\hat{R}^{-1} = \hat{R}\,\widetilde{\mathcal{I}}_2\hat{R}\,\widetilde{W}_2, \quad \widetilde{W}_2\,\hat{R}\,\widetilde{W}_2\,\hat{R} = \hat{R}\,\widetilde{W}_2\,\hat{R}\,\widetilde{W}_2, \quad \Theta_2\hat{R}^{-1}\widetilde{W}_2\hat{R} = \hat{R}\widetilde{W}_2\hat{R}\Theta_2. \end{array} \tag{3.5.58}$$

[14] In all formulas in [90], we should make the change of notation: $\omega \to \overline{\Omega}$, $J \to -\overline{\mathcal{I}}$, $L \to \overline{\mathcal{L}}$.

In the definition (3.5.57), the differential 1-forms $\overline{\Omega}{}^i{}_k$ and the inner derivatives $\overline{\mathcal{I}}{}^i{}_k$ play the role of the ghost and antighost variables. One can check directly [90] that the BRST operator $\mathsf{Q}$ given by (3.5.57) satisfies

$$\mathsf{Q}^2 = 0, \qquad [\mathsf{Q},\, \overline{\mathcal{L}}] = 0, \tag{3.5.59}$$

$$[\mathsf{Q},\, T] = T\,\overline{\Omega} \equiv d\,T, \qquad [\mathsf{Q},\, \overline{\Omega}]_+ = -\overline{\Omega}^2 \equiv d\,\overline{\Omega}, \tag{3.5.60}$$

$$[\mathsf{Q},\, \overline{\mathcal{I}}]_+ = \frac{1}{\lambda}\,(\overline{\mathcal{L}} - 1). \tag{3.5.61}$$

The (anti)commutator with $\mathsf{Q}$ (relations (3.5.60)) defines the exterior differential operator over the differential algebra (3.5.41); it provides the structure of the de Rham complex on the subalgebra with generators $T^i_j$ and $\overline{\Omega}{}^i_j$ (the de Rham complex over the quantum group $GL_q(N)$ has been firstly considered by Yu. Manin, G. Maltsiniotis, and B. Tsygan [75–77]).

To obtain relations (3.5.59)–(3.5.61), one has to use the invariance property (3.1.38) of the quantum trace $\mathrm{Tr}_{\mathcal{Q}}$ and the relations

$$\hat{R}\,\Theta_2\,\hat{R}^{-1}\,\overline{\Omega}_2 = -\overline{\Omega}_2\,\hat{R}^{-1}\,\Theta_2\,\hat{R}, \qquad \hat{R}\,\Theta_2\,\hat{R}^{-1}\,\Theta_2 = -\Theta_2\,\hat{R}^{-1}\,\Theta_2\,\hat{R}^{-1}, \tag{3.5.62}$$

$$\hat{R}^{-1}\,\Theta_2\,\hat{R}\,\overline{\mathcal{L}}_2 = \overline{\mathcal{L}}_2\,\hat{R}\,\Theta_2\,\hat{R}^{-1}, \qquad \Theta_1\,T_2 = T_2\,\hat{R}^{-1}\,\Theta_2\,\hat{R}, \tag{3.5.63}$$

$$\overline{\mathcal{I}}_2\,\hat{R}\,\Theta_2\,\hat{R}^{-1} + \hat{R}^{-1}\,\Theta_2\,\hat{R}\,\overline{\mathcal{I}}_2 = \overline{\mathcal{L}}_2\,\widetilde{W}_2^{-1}\,\hat{R}^{-1}\,\widetilde{W}_2, \tag{3.5.64}$$

which follow from (3.5.54)–(3.5.56) and (3.5.58). In particular, the condition $\mathsf{Q}^2 = 0$ follows from the last equation in (3.5.54) and equations (3.5.62) which lead to identities

$$\big(\mathrm{Tr}_{\mathcal{Q}}(\overline{\Omega})\big)^2 = 0\,, \;\; \big(\mathrm{Tr}_{\mathcal{Q}}(\Theta)\big)^2 = 0\,, \;\; [\mathrm{Tr}_{\mathcal{Q}}(\Theta),\, \mathrm{Tr}_{\mathcal{Q}}(\overline{\Omega})]_+ = 0\,. \tag{3.5.65}$$

Here we take into account that $\mathrm{Tr}_{\mathcal{Q}}(\Theta^2) = 0 = \mathrm{Tr}_{\mathcal{Q}}(\overline{\Omega}^2)$ (see Section 4 in [84]). Finally, we note (for details see [90]) that the operator $\mathsf{Q}$ given by (3.5.57) has the correct classical limit for $q \to 1$, $\lambda = q - q^{-1} \to 0$, $\hat{R}_{12} \to P_{12}$ and $\overline{\mathcal{L}} \to \mathbf{1} + \lambda X$, where elements $X^i{}_k$ are interpreted as Lie algebra generators.

**b. Quantum group covariant connections and curvatures**

To proceed further we introduce the $Z_2$-graded algebra (denoted by $\mathcal{E}$) of quantum hyperplane with generators $\{e_i, (de)_i\}$ $(i = 1, 2, \dots N)$ satisfying commutation relations

$$R_{12}e_{1\rangle}e_{2\rangle} = ce_{2\rangle}e_{1\rangle}, \quad (\pm)cR_{12}(de)_{1\rangle}e_{2\rangle} = e_{2\rangle}(de)_{1\rangle}, \quad R_{12}(de)_{1\rangle}(de)_{2\rangle} = -\tfrac{1}{c}(de)_{2\rangle}(de)_{1\rangle}. \tag{3.5.66}$$

One can recognize in these relations (for $(\pm) = +1$) the Wess–Zumino formulas of the covariant differential calculus on the bosonic $(c = q)$ and fermionic $(c = -1/q)$ quantum hyperplanes [42, 73, 74, 78, 79], where $e_i$ are the coordinates of the quantum hyperplanes and $(de)_i$ are the associated differentials (differential 1-forms). The $Z_2$-graded algebra $\mathcal{E} = \sum_{k\geqslant 0}\Omega^k(\mathcal{E})$ is the sum of subspaces $\Omega^k(\mathcal{E})$ of differential $k$-forms.

The left-coaction $\Delta_l$ of the $Z_2$-graded Hopf algebra (3.5.41) to the generators of the algebra $\mathcal{E}$ is given by the following homomorphism:

$$e_i \xrightarrow{\Delta_l} \widetilde{e}_i = T_{ij} \otimes e_j, \qquad (de)_i \xrightarrow{\Delta_l} (\widetilde{de})_i = (dT)_{ij} \otimes e_j + T_{ij} \otimes (de)_j. \tag{3.5.67}$$

The algebra $\mathcal{E}$ with generators $\{e, de\}$ becomes now a left-comodule algebra with respect to the coaction (3.5.67), since all the axioms for the comodule algebras are fulfilled [82].

Now we assume that the algebra $\mathcal{E}$ can be extended to $\bar{\mathcal{E}}$ by adding new elements $A_{ij}|_{i,j=1,\dots,N}$. We also assume that the differential $d$ can be extended onto the whole algebra $\bar{\mathcal{E}}$ and hence again this algebra is decomposed as $\bar{\mathcal{E}} = \sum_{k\geqslant 0} \Omega^k(\bar{\mathcal{E}})$. Then we postulate, first, that the elements $A_{ij}$ belong to the subspace $\Omega^1(\bar{\mathcal{E}})$ and, second, that the elements $(\nabla e)_i \in \Omega^1(\bar{\mathcal{E}})$ defined as

$$(\nabla e)_i = (de)_i - A_{ij} e_j \tag{3.5.68}$$

are transformed homogeneously under (3.5.67) as the left-comodule

$$(\nabla e)_i \xrightarrow{\Delta_l} T_{ij} \otimes (\nabla e)_j = T_{ij} \otimes ((de)_j - A_{jk} e_k)\,. \tag{3.5.69}$$

According to the classical case, we interpret the operator $A_{ij}$ satisfying (3.5.69) as a quantum deformation of a gauge potential 1-form and the operator $\nabla$ as the quantum version of the covariant derivative. The second action of $\nabla$ on both sides of (3.5.68) gives

$$(\nabla(\nabla e))_i = -\left(d(A) - A^2\right)_{ij} e_j = -F_{ij}\, e_j, \tag{3.5.70}$$

where we define the noncommutative analog of the field strength (curvature) 2-form $F$. The next action of the covariant derivative to the formula (3.5.70) yields the Bianchi identity which is written in the standard form $d(F) = [A,\ F]$. Using (3.5.67), (3.5.69), and (3.5.70), one can deduce the noncommutative analog of the gauge transformation for the noncommutative connection 1-form and curvature 2-form as

$$A_{ik} \xrightarrow{\Delta_l} \widetilde{A}_{ik} = T_{ij} T^{-1}_{lk} \otimes A_{jl} + dT_{ij} T^{-1}_{jk} \otimes 1, \qquad F_{ij} \xrightarrow{\Delta_l} \widetilde{F}_{ij} = \left(T_{ik} T^{-1}_{lj}\right) \otimes F_{kl}. \tag{3.5.71}$$

As it was argued in [82, 83], the possible choice of the covariant algebra of the connection 1-form $A$ and curvature 2-form $F$ is given by the defining relations

$$F_1 \hat{R}_{12} A_1 \hat{R}_{12} = \hat{R}_{12} A_1 \hat{R}_{12} F_1, \qquad F_1 \hat{R}_{12} F_1 \hat{R}_{12} = \hat{R}_{12} F_1 \hat{R}_{12} F_1, \tag{3.5.72}$$

$$\hat{R}_{12} A_1 \hat{R}_{12} A_1 + A_1 \hat{R}_{12} A_1 \hat{R}^{-1}_{12} = \lambda\, g\, (\hat{R}_{12} F_1 + F_1 \hat{R}^{-1}_{12}), \tag{3.5.73}$$

where $\lambda = q - q^{-1}$ and $g$ is an arbitrary parameter. In particular, to check the commutation relations (3.5.73) for the elements $A_{ij}$, we remark that there is a representation for the generators $A_{ij}$, namely $A = dTT^{-1} \otimes 1$, which is related to the flat connection $F_{ij} = 0$. Using this representation and formulas (3.5.41), we conclude that the generators $A_{ij}$ have to satisfy relation (3.5.73) with the r.h.s. equal to zero. In what follows, we consider only the case $g = 0$. Note that the algebra (3.5.72), (3.5.73) (for $g = 0$) is covariant not only under coaction (3.5.71) of the $RTT$ algebra, but also is a braided comodule algebra with respect to the braided coaction of the $RLRL$ algebra [88].

Let $\hat{R}$ be a skew-invertible $R$-matrix for which we define the quantum traces (3.1.39) with properties (3.1.38) (see also (3.2.12), (3.2.13), and (3.2.14)). By analogy with the classical case, we can consider the noncommutative version of the invariant Chern characters [82, 83, 88]:

$$C^{(k)} = \mathrm{Tr}_q(F^k) = \mathrm{Tr}(DF^k) = D_{ij} F_{jj_1} F_{j_1 j_2} \dots F_{j_{k-1} i}, \tag{3.5.74}$$

where we have used the quantum trace (3.2.12), with matrix $D$. Chern characters (3.5.74) are central elements for the algebra (3.5.72) (the proof is the same as proof of (3.2.34)). Applying (3.2.13), we immediately obtain that $2k$-forms $C^{(k)}$ (3.5.74) are coinvariants under the adjoint cotransformation of $F$ given in (3.5.71). Moreover, $C^{(k)}$ are the closed $2k$-forms. Indeed, from the Bianchi identities $dF = [A, F]$ we deduce

$$dC^{(k)} = \mathrm{Tr}_q(A\,F^k - F^k\,A) = 0, \tag{3.5.75}$$

where we have taken into account (see Eqs. (3.1.22), (3.1.36), (3.5.72)):

$$\begin{gathered}\mathrm{Tr}_q(A\,F^k) = \mathrm{Tr}_{q1}\mathrm{Tr}_{q2}(\hat{R}_{12}^{-1}\,\hat{R}_{12}\,A_1\,\hat{R}_{12}\,F_1^k) = \mathrm{Tr}_{q1}\mathrm{Tr}_{q2}(\hat{R}_{12}^{-1}\,F_1^k\,\hat{R}_{12}\,A_1\,\hat{R}_{12}) = \\ = \mathrm{Tr}_{q1}\mathrm{Tr}_{q2}(F_1^k\,\hat{R}_{12}\,A_1) = \mathrm{Tr}_q(F^k\,A).\end{gathered} \tag{3.5.76}$$

We note that $\mathrm{Tr}_q(A^{2k}) = 0$ for the algebra (3.5.73), when $g = 0$ (see Proposition 4 in [84]). In view of this, the natural conjecture is that $C^{(k)}$ have to be presented, for $g = 0$, as the exact form $C^{(k)} = dL^{(k)}_{\mathrm{CS}}$, where the noncommutative Chern–Simons $(2k-1)$-forms $L^{(k)}_{\mathrm{CS}}$ are

$$L^{(k)}_{\mathrm{CS}} = \mathrm{Tr}_q\Big(A(dA)^{k-1} + \frac{1}{h_1^{(k)}}A^3(dA)^{k-2} + \cdots + \frac{1}{h_{k-1}^{(k)}}A^{2k-1}\Big), \tag{3.5.77}$$

and unknown coefficients $h_j^{(k)}$ depend on the choice of the Hecke matrix $\hat{R}$ (in the classical case $\hat{R}_{12} = P_{12}$ and $q = 1$, all these coefficients are known [92]). We checked this conjecture in the case $k = 2$, for $GL_q(N)$ $R$-matrix (3.4.8) and the special algebra (3.5.73), when $g = 0$. In this case, we obtained [82, 83] a noncommutative analog of the three-dimensional Chern–Simons term in the form

$$L^{(2)}_{\mathrm{CS}} = \mathrm{Tr}_q\Big(A\,dA + \frac{1}{h_1^{(2)}}A^3\Big), \qquad h_1^{(2)} = -\,\frac{q^2+1+q^{-2}}{q^2+q^{-2}}. \tag{3.5.78}$$

**Remark.** The elements of the differential calculus on the $RLRL$ (reflection equation) algebra were considered in papers [23–25, 88], [93] (see also references therein).

*3.5.4. α-Deformation of the Heisenberg double of RTT and RLRL algebras. Quantum Cayley–Hamilton–Newton identities*

Now for the right HD (3.2.57) (the algebra (3.5.26) with upper sign) we calculate the commutation relations of the elements $a_m(L)$ with generators $T^i_j$ of the $RTT$ algebra defined by the Hecke-type $R$-matrix. Note that in the case of the Heisenberg double of $\mathrm{Fun}(SL_q(N))$ and $U_q(sl(N))$, we need to renormalize the Hecke $R$-matrix: $\hat{R} \to q^{-1/N}\hat{R}$ according to (3.4.40). This leads to the following generalization of the cross-multiplication rules (3.2.57) (we consider only the right HD):

$$T_1 L_2 = \alpha\,\hat{R}_{12}L_1\hat{R}_{12}T_1, \tag{3.5.79}$$

where $\hat{R}$ is a Hecke $R$-matrix (3.4.11) of the height $N$ and for the special case of the $SL_q(N)$-type HD we have to fix $\alpha = q^{-2/N}$ (but generally the constant $\alpha \neq 0$ is arbitrary). So, the commutation relations (3.5.79) define the one-parameter deformation of the Heisenberg double of $RTT$ and $RLRL$ algebras for the Hecke-type $R$-matrix. Note that the automorphism (3.2.60) is only correct for the choice $\alpha = 1$ in (3.5.79). For example, in view of (3.5.79), the quantum matrices $(L + x)T$ start to obey the modified $RTT$ relations

$$\hat{R}_1\,(L_1 + x)T_1\;(\alpha^{-1}L_2 + x)T_2 = (L_1 + x)T_1\;(\alpha^{-1}L_2 + x)T_2\,\hat{R}_1. \tag{3.5.80}$$

However, for the general choice of $\alpha$ in (3.5.79), the definition of the characteristic polynomial (3.5.34) is not changed, since instead of (3.5.33), we can take

$$\begin{aligned}(L_1+x)T_1\,(\alpha^{-1}L_2+x)T_2\dots(\alpha^{1-N}L_N+x)T_N\,\mathcal{E}^{1\dots N\rangle}\,\mathrm{det}_q^{-1}(T) = \\ = \big((L_{\widetilde{1}}+x)\,(L_{\widetilde{2}}+x)\dots(L_{\widetilde{N}}+x)\big)\,\mathcal{E}^{1\dots N\rangle} = \mathcal{E}^{1\dots N\rangle}\,\mathrm{Det}_q(L;x)\end{aligned} \tag{3.5.81}$$

(according to (3.5.80), we modify the first line in (3.5.33) but it does not affect the final expressions for $\mathrm{Det}_q(L;x)$ and coefficients $a_m(L)$). To calculate the commutation relations of $a_m(L)$ with $T^i_j$, we find (by using (3.5.81), (3.5.80), (3.5.22), and (3.5.25))

$$\begin{aligned}&(L_1+x)T_1\,\mathrm{Det}_q(L;\alpha x)\,\mathcal{E}^{2\dots N+1\rangle} = \\ &= (L_1+x)T_1\,(\alpha^{-1}L_2+x)T_2\dots(\alpha^{-N}L_{N+1}+x)T_{N+1}\,\mathcal{E}^{2\dots N+1\rangle}\,\tfrac{\alpha^N}{\mathrm{det}_q(T)} = \\ &= \hat{R}_1\dots\hat{R}_N(L_1+x)T_1\dots(\alpha^{-N}L_{N+1}+x)T_{N+1}\,\hat{R}_N^{-1}\dots\hat{R}_1^{-1}\,\mathcal{E}^{2\dots N+1\rangle}\,\tfrac{\alpha^N}{\mathrm{det}_q(T)} = \\ &= \mathcal{E}^{2\dots N+1\rangle}N^{1\rangle}_{\langle N+1}\,\mathrm{Det}_q(L;x)\,\mathrm{det}_q(T)\big(\alpha^{-N}L_{N+1}+x\big)T_{N+1}\,(N^{-1})^{N+1\rangle}_{\langle 1}\,\tfrac{\alpha^N}{\mathrm{det}_q(T)} = \\ &= \alpha^N\mathcal{E}^{2\dots N+1\rangle}N^{1\rangle}_{\langle N+1}\,\mathrm{Det}_q(L;x)\,\big(q^2(N^{-1}LN)_{_{N+1}}+x\big)(N^{-1}TN)_{_{N+1}}\,(N^{-1})^{N+1\rangle}_{\langle 1} = \\ &= \alpha^N\,\mathcal{E}^{2\dots N+1\rangle}\,\mathrm{Det}_q(L;x)\,(q^2\,L_1+x)\,T_1,\end{aligned} \tag{3.5.82}$$

where we have taken into account the commutation relations of $\mathrm{det}_q(T)$ and $L^i_j$ deduced by the standard method:

$$\begin{aligned}\mathcal{E}_{\langle 1\dots N}\,\mathrm{det}_q(T)\,L_{N+1} = \mathcal{E}_{\langle 1\dots N}\,T_1\dots T_N\,L_{N+1} = \\ = \alpha^N\,\mathcal{E}_{\langle 1\dots N}\,\hat{R}_N\dots\hat{R}_1L_1\hat{R}_1\dots\hat{R}_NT_1\dots T_N = q^2\,\alpha^N\,(N^{-1}\,L\,N)_{_{N+1}}\,\mathrm{det}_q(T)\,\mathcal{E}_{\langle 1\dots N}\end{aligned} \tag{3.5.83}$$

(Eqs. (3.5.79) and (3.5.20) were applied). Thus, we have the following relations (see (3.5.82)):

$$(L_1+x)T_1\,\mathrm{Det}_q(L;\alpha x) = \alpha^N\,\mathrm{Det}_q(L;x)\,(q^2\,L_1+x)\,T_1. \tag{3.5.84}$$

The expansion of (3.5.84) over $x$ gives the recurrent equation for desired commutation relations of $a_k(L)$ with $T^i_j$ ($k \geqslant 0$):

$$\alpha^{-k}\,LT\,a_k + \alpha^{-1-k}\,T\,a_{k+1} = q^2\,a_k\,LT + a_{k+1}T, \quad T\,a_0 = a_0\,T.$$

These equations are easy to solve by iteration, and the solution is

$$\alpha^{-k}\,Ta_k = a_kT - (q^2-1)\sum_{m=1}^{k}(-1)^m a_{k-m}L^m\,T. \tag{3.5.85}$$

Since the matrix $T$ is invertible, we write this equation in the form

$$\alpha^{-k}\,Ta_k\,T^{-1} = a_k - (q^2-1)\sum_{m=1}^{k}(-1)^m a_{k-m}L^m. \tag{3.5.86}$$

For the left-hand side of (3.5.86), by using the definition (3.5.40) of $a_k$, we deduce

$$\begin{aligned}\alpha^{-k}\,q^{-k}\,T_1a_k\,T_1^{-1} = \alpha^{-k}\,T_1\,\mathrm{Tr}_{D(2\dots k+1)}\Big(A_{2\dots k+1}\,L_2\hat{R}_2L_2\hat{R}_2^{-1}\dots\hat{R}_{k\leftarrow 2}L_2\hat{R}_{k\leftarrow 2}^{-1}\Big)\,T_1^{-1} = \\ = \mathrm{Tr}_{D(2\dots k+1)}\Big(A_{2\dots k+1}\,\hat{R}_1\dots\hat{R}_k\,L_{\underline{1}}\dots L_{\underline{k}}\,\hat{R}_k\dots\hat{R}_1\,A_{2\dots k+1}\Big) = \\ = \mathrm{Tr}_{D(2\dots k+1)}\Big(\hat{R}_{(1\to k)}\,A_{1\dots k}\,L_{\underline{1}}\dots L_{\underline{k}}\,A_{1\dots k}\,\hat{R}_{(k\leftarrow 1)}\Big) = \\ = \mathrm{Tr}_{D(2\dots k+1)}\Big(\hat{R}_1\dots\hat{R}_k\,A_{1\dots k}\,L_{\underline{1}}\dots L_{\underline{k}}\,A_{1\dots k}(\hat{R}_k^{-1}+\lambda)\,\hat{R}_{(k-1\leftarrow 1)}\Big) =\end{aligned}$$

$$= \mathrm{Tr}_{D(2\dots k)} \Big( \hat{R}_{(1\to k-1)} \left[ \mathrm{Tr}_{D(k)}(A_{1\dots k} L_{\underline{1}} \dots L_{\underline{k}} \, A_{1\dots k}) \right] \hat{R}_{(k-1\leftarrow 1)} \Big) + \\ + \lambda \, q^{2(1-k)} \, \mathrm{Tr}_{D(2\dots k)} \left( A_{1\dots k} L_{\underline{1}} \dots L_{\underline{k}} \right), \tag{3.5.87}$$

where in the last transformation we apply the first identities in (3.1.22) and (3.1.38). By repeating this transformation (3.5.87) for many times, we obtain

$$\alpha^{-k} \, q^{-k} \, T_1 a_k \, T_1^{-1} = q^{-k} a_k + \lambda (1 + q^{-2} + \dots q^{2(1-k)}) \mathrm{Tr}_{D(2\dots k)} \left( A_{1\dots k} L_{\underline{1}} \dots L_{\underline{k}} \right) = \\ = q^{-k} a_k + q(1 - q^{-2k}) \mathrm{Tr}_{D(2\dots k)} \left( A_{1\dots k} L_{\underline{1}} \dots L_{\underline{k}} \right). \tag{3.5.88}$$

Comparing (3.5.86) and (3.5.88), we obtain the remarkable identities for quantum RE matrices $L$ (the so-called Cayley–Hamilton–Newton identities [125]):

$$[k]_q \, \mathrm{Tr}_{D(2\dots k)}(A_{1\to k} L_{\underline{1}} \dots L_{\underline{k}}) = - \sum_{m=1}^{k} (-1)^m a_{k-m} L_{\underline{1}}^m. \tag{3.5.89}$$

It follows from (3.5.89) (apply $\mathrm{Tr}_{D(1)}$ to the both sides) that the two basic sets (3.2.29), (3.5.40) of central elements for the RE algebra (defined by the Hecke-type $R$-matrix) are related by the $q$-analogue of the Newton relations:

$$\frac{[k]_q}{q^k} \, a_k + \sum_{m=1}^{k} (-1)^m a_{k-m} \, p_m = 0, \qquad k = 1, \dots, N, \tag{3.5.90}$$

where we introduce power sums $p_m = \mathrm{Tr}_D(L^m)$, $m = 1, \dots, N$, and we imply $a_0 = 1$. Note that in view of (3.5.30), (3.5.38), and (3.5.8), we have

$$[N]_q \, \mathrm{Tr}_{D(2\dots N)}(A_{1\to N} L_{\underline{1}} \dots L_{\underline{N}}) = [N]_q \, \mathrm{Tr}_{D(2\dots N)} \big( A_{1\to N} L_{\widetilde{1}} \dots L_{\widetilde{N}} (y_2 \cdots y_N)^{-1} \big) = \\ = [N]_q \, \mathrm{Det}_q(L) \, q^{(N-1)N} \, \mathrm{Tr}_{D(2\dots N)}(A_{1\to N}) = \mathrm{Det}_q(L) \, I_1 \equiv a_N \, I_1 .$$

Thus, for $k = N$ the relation (3.5.89) provides the characteristic identity for the quantum matrix $L$ ($q$-analogue of the Cayley–Hamilton theorem):

$$\sum_{k=0}^{N} (-L)^k \, a_{N-k}(L) = 0. \tag{3.5.91}$$

This identity can formally be obtained by the substitution of $x = -L$ in the characteristic polynomial (3.5.34). Therefore, in view of (3.5.90) and (3.5.91), the elements $a_m(L)$ can be interpreted as noncommutative analogs of elementary symmetric functions for eigenvalues of the quantum matrix $L$ (see details in [129]).

Introduce generating functions $a(t), p(t)$ for elementary symmetric functions and power sums:

$$a(t) = \sum_{k \geqslant 0} a_k \, t^k, \quad p(t) = \sum_{k \geqslant 1} p_k \, t^k.$$

Then it is worth noting [47] that quantum Newton relations (3.5.90) can be written as a finite difference equation for $a(t)$:

$$a(t) \, p(-t) = \frac{a(q^{-2} t) - a(t)}{q - q^{-1}}. \tag{3.5.92}$$

This equation shows that the power sums $p_k$ can always be expressed as polynomials of the elementary symmetric functions $a_k$.

The Cayley–Hamilton–Newton identities (3.5.89) for the $GL(n)$-type quantum matrix algebra were invented in [125]. It seems that these matrix identities were unknown even for the case of usual commutative matrices ($q = 1$). For the reflection equation algebra, in the case $N = 2$, the identity (3.5.91) was considered in [141] and in [264]. For general $N$ these identities were proved in [126, 127]. The Newton relations (3.5.90) have been obtained in [127, 128]. Identities (3.5.82)–(3.5.85) and their special cases were essentially used in [129] (see Propositions 3.21 and 3.24 there). Moreover, the spectral properties of the reflection equation matrices $L^i_j$ were investigated in [129]. In fact, all algebraic relations and identities of this section were important for the investigations [129] of the theory of the $q$-deformed isotropic top [98, 124].

For $GL(m|n)$-type quantum supermatrix algebras, Cayley–Hamilton identities were obtained in [130]. For orthogonal and symplectic types quantum matrix algebras, Cayley–Hamilton identities and Newton relations were derived in [131].

### *3.6. Multiparameter deformations of linear groups*

In this subsection, we consider a multiparameter deformation of the linear group $GL(N)$ (see [57, 74] and [132–137]). A multiparameter quantum hyperplane is defined by the relations

$$x^i x^j = r_{ij} x^j x^i, \quad i < j, \tag{3.6.1}$$

which can be written in the $R$-matrix form (3.4.6) if we introduce an additional parameter $q$. Thus, we have $N(N-1)/2 + 1$ deformation parameters: $r_{ij}$, $i < j$ and $q$. The corresponding $R$-matrix is (see, e.g., [57])

$$R_{12} = q \sum_i e_{i,i} \otimes e_{i,i} + \sum_{i \neq j} (e_{i,i} \otimes e_{j,j}) a_{ji} + (q - q^{-1}) \sum_{i>j} e_{i,j} \otimes e_{j,i}, \tag{3.6.2}$$

where $a_{ij} = 1/a_{ji} = r_{ij}/q$ (for $i > j$), and it can be represented in components as

$$R^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \left( q\delta^{i_1 i_2} + \Theta_{i_2 i_1} \frac{q}{r_{i_1 i_2}} + \Theta_{i_1 i_2} \frac{r_{i_2 i_1}}{q} \right) + (q - q^{-1}) \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \Theta_{i_1 i_2}, \tag{3.6.3}$$

where $\Theta_{ij}$ is defined in (3.4.10). The $R$-matrix (3.6.2) is obtained by the twisting of the standard one-parameter $R$-matrix (3.4.8) (see Subsection **2.5** and Eqs. (2.5.6), (3.2.68)):

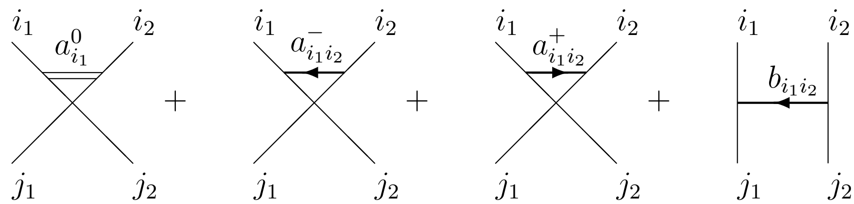
$$R_{12} \to F_{21} R_{12} F_{12}^{-1} \Leftrightarrow \hat{R}_{12} \to F_{12} \hat{R}_{12} F_{12}^{-1}, \quad F_{12} = \sum_{i,j} (e_{i,i} \otimes e_{j,j}) \, f_{ij}, \tag{3.6.4}$$

where $a_{ij} = f_{ij}/f_{ji}$ and $\hat{F} = PF$ satisfies the twisting matrix conditions (3.2.67). Thus, the multiparametric $R$-matrix (3.6.3) is reduced to the one-parameter $R$-matrix with the help of the appropriate twisting (see also [57] and [137]).

By the construction, in view of the twisting procedure (3.6.4), (3.2.67), the $R$-matrix (3.6.3) satisfies the Yang–Baxter equation (3.1.11) and the same Hecke condition (3.4.11) as in the one-parameter case.

Now, to justify expression (3.6.3), we try to find the most general Yang–Baxter solution $R_{12}$ of the form (3.4.7). We only require that the $R$-matrix (3.4.7) has the lower-triangular block form: $b_{ij} = 0$ for $i \geqslant j$ (as it was shown in [68], this condition is not restrictive). When we check the fulfillment of the Yang–Baxter equation, it is convenient to use the diagrammatic technique [46]:

$$\hat{R} = \hat{R}^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\left(a^0_{i_1}\delta^{i_1 i_2} + \Theta_{i_2 i_1}a^-_{i_1 i_2} + \Theta_{i_1 i_2}a^+_{i_1 i_2}\right) + b_{i_1 i_2}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\Theta_{i_2 i_1} = \tag{3.6.5}$$




It turns out that not all solutions of the Yang–Baxter equation (3.1.11) that can be represented in the form (3.6.5) are exhausted by the multiparameter $R$-matrices (3.6.3). Indeed, if we substitute the matrix (3.6.5) into the Yang–Baxter equation (3.1.11), we obtain the following general conditions on the coefficients $a^0_i$, $a^\pm_{ij}$, $b_{ij}$:

$$b_{ij} = b, \;\; a^+_{ij}a^-_{ji} = c, \;\; (a^0_i)^2 - b a^0_i - c = 0 \;\; (\forall i,j). \tag{3.6.6}$$

We normalize (3.6.5) in such a way that $c = 1$ and choose for convenience, instead of the parameter $b$, a different parameter $q$, setting $b = q - q^{-1}$. Then $a^0_i$ can take two values $\pm q^{\pm 1}$. For such a normalization, the solution of the Yang–Baxter equation of the form (3.6.5) automatically satisfies the Hecke relation (3.4.11). If we set $a^0_i = q$ (or $a^0_i = -q^{-1}$) for all $i$, then we arrive at the many-parametric case $GL_{q,r_{ij}}(N)$ (3.6.3) (up to exchange $q \to -q^{-1}$ in the case $a^0_i = -q^{-1}$). If, however, we set

$$a^0_i = q \; (1 \leqslant i \leqslant M), \qquad a^0_i = -q^{-1} \;\; (M+1 \leqslant i \leqslant N), \tag{3.6.7}$$

then the $R$-matrix (3.6.5) does not reduce to (3.6.3) and will correspond to a multi-parameter deformation of the supergroup $GL(M|N-M)$:

$$\hat{R}_{12} = \sum_i (-1)^{[i]}\, q^{1-2[i]}\, e_{ii}\otimes e_{ii} + \sum_{i\neq j} a^+_{ij}\, e_{ij}\otimes e_{ji} + \lambda \sum_{j>i} e_{ii}\otimes e_{jj}, \tag{3.6.8}$$

where $i,j = 1,\dots,N+M$, $[i] = 0,1$ (mod(2)), we take into account (3.6.7) and $a^+_{ij} = 1/a^+_{ji}$ for $i > j$. We consider this case (for a special choice of $a^+_{ij}$) below in Subsection **3.7**.

By virtue of the fulfillment of the Hecke identity (3.4.11) for the multiparameter case, we can introduce the same projectors $\mathbf{P}^-$ and $\mathbf{P}^+$ as in the one-parameter case (3.4.21); the first of them defines the bosonic quantum hyperplane (3.6.1) (the relations (3.4.6) with $R$-matrix (3.6.3)), and the second one defines the fermionic quantum hyperplane:

$$\mathbf{P}^+ x_1 x_2 = 0 \qquad \Leftrightarrow \qquad (x^i)^2 = 0, \quad q^2 x^i x^j = -r_{ij} x^j x^i \; (i > j). \tag{3.6.9}$$

Regarding (3.6.1) and (3.6.9) as comodules for the multiparameter quantum group $GL_{q,r_{ij}}(N)$, we find that the generators $T^i_j$ of the algebra $\mathrm{Fun}(GL_{q,r_{ij}}(N))$ satisfy the same $RTT$ relations (3.2.1) but with $R$-matrix (3.6.3). Note, however, that the quantum determinant $\det_q(T)$ (3.4.30) is not central in the multiparameter case [135]. This is due to the fact that in general for the multiparameter $R$-matrix we have $N \neq \mathrm{const}\cdot I$ in Eqs. (3.5.21) and (3.5.25). Therefore, reduction to the $SL$ case by means of the condition $\det_q(T) = 1$ is possible only under certain restrictions on the parameters $q, r_{ij}$. A detailed discussion of these facts can be found in [135, 137].

The algebra (3.2.21), (3.2.22) (with the multiparameter $R$-matrix (3.8.3)) which is dual to the algebra $\mathrm{Fun}(GL_{q,r_{ij}}(N))$ can also be considered. It appears that this algebra is isomorphic to the one-parameter deformation of $gl(N)$ (3.4.54)–(3.4.58). One can find details about the dual algebras for the special case of $\mathrm{Fun}(GL_{q,p}(2))$ in papers [141, 142].

### *3.7. The quantum supergroups $GL_q(N|M)$ and $SL_q(N|M)$*

We choose the Hecke-type $R$-matrix (3.6.5), (3.6.8) and write it in the form (cf. [150–152])

$$\hat{R} = \sum_i (-1)^{[i]}\, q^{1-2[i]}\, e_{ii} \otimes e_{ii} + \sum_{i\neq j} (-1)^{[i][j]}\, e_{ij} \otimes e_{ji} + \lambda \sum_{j>i} e_{ii} \otimes e_{jj}, \tag{3.7.1}$$

where we have set (see (3.6.8))

$$a^0_i = (-1)^{[i]} q^{1-2[i]}, \;\; a^+_{ij} = (a^-_{ij})^{-1} = (-1)^{[i][j]}, \;\; b = q - q^{-1} = \lambda.$$

We stress here that the matrix units $e_{ij}$ and tensor products in (3.7.1) *are not graded*, as follows form the previous Subsection **3.6**. The component presentation of (3.7.1) is

$$\hat{R}^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\, (-1)^{[i_1][i_2]}\, q^{\delta_{i_1 i_2}\,(1-2[i_1])} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\, \lambda\, \Theta_{i_2 i_1}. \tag{3.7.2}$$

Thus, the parameters $a^0_i$ take the two values $\pm q^{\pm 1}$ and, as we assumed it in Subsection **3.6**, the $R$-matrix (3.7.1), (3.7.2) must correspond to some supergroup. Indeed, in the limit $q \to 1$, we find that $\hat{R}$ tends to the supertransposition operator

$$\hat{R}^{i_1 i_2}_{j_1 j_2} \to (-1)^{[i_1][i_2]} \delta^{i_1}_{j_2}\delta^{i_2}_{j_1} \equiv \mathcal{P}_{12}. \tag{3.7.3}$$

Suppose that the $R$-matrix acts in the space of the direct product $x \underline{\otimes}\, y$ of two supervectors $x$ and $y$ with coordinates $x^{j_1}$ and $y^{j_2}$, and $[i] = 0, 1$ denotes the parity (grading) of the components[15] $x^i$ and $y^i$. According to (3.7.3), we write the condition for the graded tensor product $\underline{\otimes}$ as

$$x^{j_1} \underline{\otimes}\, y^{j_2} = \mathcal{P}^{j_1 j_2}_{k_1 k_2} (1 \underline{\otimes}\, y^{k_1})(x^{k_2} \underline{\otimes}\, 1) \quad \Rightarrow \quad x^{j_1} \underline{\otimes}\, y^{j_2} = (-1)^{[j_1][j_2]} (1 \underline{\otimes}\, y^{j_2})(x^{j_1} \underline{\otimes}\, 1).$$

For definiteness, we will assume that

$$[i] = 0 \quad (1 \leqslant i \leqslant N), \quad [i] = 1 \quad (N+1 \leqslant i \leqslant N+M). \tag{3.7.4}$$

As we noted in Subsection **3.6**, the $R$-matrix (3.7.2) satisfies the Yang–Baxter equation (3.1.11) (in the braid group form) and the Hecke relation (3.1.68). In addition to the matrix $\hat{R}$, we introduce the new $R$-matrix:

$$R_{12} = \mathcal{P}_{12} \hat{R}_{12} = (-)^{(1)(2)} P_{12} \hat{R}_{12} =$$

$$= \sum_i q^{1-2[i]}\, e_{ii} \otimes e_{ii} + \sum_{i\neq j} e_{ii} \otimes e_{jj} + \lambda \sum_{i>j} (-1)^{[i][j]}\, e_{ij} \otimes e_{ji} \tag{3.7.5}$$

[15] There are two equivalent descriptions of supervector spaces $V$. The first one is to consider the graded basis vectors $e_i$, while coordinates $x^i$ of supervectors $e_i x^i \in V$ are ordinary numbers. Another (dual) approach is that vectors $e_i$ form a bases of an ordinary vector space, but coordinates $x^i$ are graded in such a way that $e_i x^i$ belongs to the superspace $V$. Here we use the second approach.

with the semiclassical behavior (3.3.1). Here and below we use notation

$$\big((-)^{(1)(2)}\big)^{i_1 i_2}_{j_1 j_2} := (-1)^{[i_1][i_2]}\, \delta^{i_1}_{j_1}\, \delta^{i_2}_{j_2}. \qquad (3.7.6)$$

Then, for the new $R$-matrix (3.7.5), we obtain from Eq. (3.1.11) the graded form [154] of the Yang–Baxter equation:

$$R_{12}(-)^{(2)(3)} R_{13}(-)^{(2)(3)} R_{23} = R_{23}(-)^{(2)(3)} R_{13}(-)^{(2)(3)} R_{12}, \qquad (3.7.7)$$

where we have taken into account the fact that $R_{12}$ is an even $R$-matrix, i.e.,

$$R^{i_1 i_2}_{j_1 j_2} \neq 0 \;\text{ if }\; [i_1] + [j_1] + [i_2] + [j_2] = 0 \; (\mathrm{mod}(2)) \;\Rightarrow$$
$$(-1)^{[i_3]([i_1]+[i_2])} R^{i_1 i_2}_{j_1 j_2} \delta^{i_3}_{j_3} = R^{i_1 i_2}_{j_1 j_2} \delta^{i_3}_{j_3}\, (-1)^{[j_3]([j_1]+[j_2])} \;\Leftrightarrow$$
$$(-)^{(3)((1)+(2))}\, R_{12}\, I_3 = R_{12}\, I_3\, (-)^{(3)((1)+(2))}.$$

In the last relation, we set

$$\big((-)^{(3)((1)+(2))}\big)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = (-1)^{[i_3]([i_1]+[i_2])}\, \delta^{i_1}_{j_1}\, \delta^{i_2}_{j_2}\, \delta^{i_3}_{j_3}.$$

For the $R$-matrix (3.7.1), (3.7.2) we will also use the properties

$$\hat{R}_{12} = (-)^{(1)(2)} \hat{R}_{12} (-)^{(1)(2)}, \;\; (-)^{(1)+(2)} \hat{R}_{12} = \hat{R}_{12} (-)^{(1)+(2)}. \qquad (3.7.8)$$

Since $GL_q(N|M)$ $R$-matrix (3.7.2) satisfies the Hecke condition, we find

$$(\hat{R}^{-1})^{i_1,i_2}_{j_1,j_2} = \hat{R}^{i_1,i_2}_{j_1,j_2} - \lambda\, \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}\, (-1)^{[i_1][i_2]}\, q^{\delta_{i_1 i_2}\, (2[i_1]-1)} - \lambda\, \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}\, \Theta_{i_1 i_2},$$

and we have the identities (cf. (3.4.12))

$$\hat{R}^{-1}_{12}[q^{-1}] = \hat{R}_{21}[q]. \qquad (3.7.9)$$

Finally, the skew-inverse matrix $\Psi_{12}$ (3.1.18) for the $GL_q(N|M)$ $R$-matrix, defined in (3.7.1), (3.7.2), has the form

$$\begin{gathered}\hat{\Psi}_{12} = \sum_i e_{ii} \otimes e_{ii} (-1)^{[i]} q^{2[i]-1} + \sum_{i \neq j} (-1)^{[i][j]} e_{ij} \otimes e_{ji} - \\ -\lambda \sum_{i<j} e_{ii} \otimes e_{jj} (-1)^{[i]+[j]} q^{(-1)^{[i]} (2i-2N-1) - (-1)^{[j]} (2j-2N-1)}, \\ \hat{\Psi}^{i_1 i_2}_{j_1 j_2} = (-1)^{[i_1][i_2]}\, q^{\delta_{i_1 i_2}\, (2[i_1]-1)}\, \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} - \\ -(-1)^{[i_1]+[i_2]} \lambda\, q^{(-1)^{[i_1]} (2i_1 - 2N - 1)}\, q^{(-1)^{[i_2]} (1+2N-2i_2))}\, \Theta_{i_2 i_1}\, \delta^{i_1}_{j_1} \delta^{i_2}_{j_2},\end{gathered} \qquad (3.7.10)$$

which follows from the general formula (3.4.14) (for the case $[i_1] = [i_2] = 0$, we reproduce the matrix (3.4.15)). The corresponding matrices of quantum supertraces are

$$\begin{gathered}D_1 \equiv \mathrm{Tr}_2 \left(\hat{\Psi}_{12}\right) \;\; \Rightarrow \;\; D^i_j = (-1)^{[i]}\, q^{2M + (-1)^{[i]} (2i-2N-1)}\, \delta^i_j, \\ Q_2 \equiv \mathrm{Tr}_1 \left(\hat{\Psi}_{12}\right) \;\; \Rightarrow \;\; Q^i_j = (-1)^{[i]}\, q^{-2M + (-1)^{[i]} (2N+1-2i)} \delta^i_j, \\ \mathrm{Tr}(D) = \mathrm{Tr}(Q) = (1 - q^{2(M-N)})/\lambda = q^{(M-N)}\, [N-M]_q.\end{gathered} \qquad (3.7.11)$$

Note that the quantum supertrace $\mathrm{Tr}_D$, which is constructed by means of the matrix $D$ (see the first line in (3.7.11)), coincides up to the factor $q^{(3M-N)/2}$ with the quantum trace presented in [141]. For $q \to 1$ quantum supertraces $\mathrm{Tr}_{_D}$ and $\mathrm{Tr}_{_Q}$ tend to the usual supertraces.

The quantum multidimensional superplanes for the Hecke-type $R$-matrix (3.7.2) are defined as algebras $\mathcal{V}_\pm$ with generators $x_i$ ($i = 1, \dots, N+M$) and defining relations (see, for example, [74, 153] and [141]):

$$\begin{aligned} &\mathcal{V}_- : \ (\hat{R} - q)x^{1\rangle}x^{2\rangle} = 0 \quad \Leftrightarrow \quad x^i x^j = (-1)^{[i][j]} q x^j x^i \ \ (i<j), \ (x^i)^2 = 0 \text{ if } [i]=1, \\ &\mathcal{V}_+ : \ (\hat{R} + q^{-1})x^{1\rangle}x^{2\rangle} = 0 \ \Leftrightarrow \ q x^i x^j = -(-1)^{[i][j]} x^j x^i \ \ (i<j), \ (x^i)^2 = 0 \text{ if } [i]=0. \end{aligned} \tag{3.7.12}$$

The super-hyperplane $\mathcal{V}_+$ can be interpreted (see, e.g., [84]) as an exterior algebra of differentials $dx^i$ of the coordinates $x^i$ for the first hyperplane $\mathcal{V}_-$.

We take the left coaction (3.4.3) of the quantum supergroup with generators $T^i_j$ to the quantum superspaces $\mathcal{V}_\pm$, defined in (3.7.12), and consider this coaction to the spaces $\mathcal{V}_\pm \underline{\otimes} \mathcal{V}_\pm$:

$$(T^{i_1}_{j_1} \underline{\otimes}\, x^{j_1})\,(T^{i_2}_{j_2} \underline{\otimes}\, x^{j_2}) = (-1)^{[j_1]([i_2]+[j_2])}\,(T^{i_1}_{j_1}\, T^{i_2}_{j_2}) \underline{\otimes}\, (x^{j_1}\, x^{j_2}), \tag{3.7.13}$$

where $\underline{\otimes}$ is understood as a graded direct product. We postulate the gradings of the elements $T^i_j$ and $x^i$ as $[T^i_j] = [i]+[j]$ and $[x^i] = [i]$. In [154], the right coaction of the quantum supergroup was considered with another signs in the formulas, but it can be shown that this difference is not essential.

From the condition of covariance of the relations (3.7.12) under coaction (3.7.13), we deduce the graded form of the $RTT$ equations:

$$\hat{R}^{i_1 i_2}_{k_1 k_2}\, T^{k_1}_{j_1}\, (-1)^{[j_1][k_2]}\, T^{k_2}_{j_2}\, (-1)^{[j_1][j_2]} = T^{i_1}_{k_1}\, (-1)^{[k_1][i_2]}\, T^{i_2}_{k_2}\, (-1)^{[k_1][k_2]}\, \hat{R}^{k_1 k_2}_{j_1 j_2}, \tag{3.7.14}$$

written, with the help of the concise matrix notation (3.7.6), as (cf. (3.2.66))

$$\begin{aligned} \hat{R} T_1 (-)^{(1)(2)} T_2 (-)^{(1)(2)} = T_1 (-)^{(1)(2)} T_2 (-)^{(1)(2)} \hat{R} \Leftrightarrow \\ R_{12} T_1 (-)^{(1)(2)} T_2 (-)^{(1)(2)} = (-)^{(1)(2)} T_2 (-)^{(1)(2)} T_1 R_{12}, \end{aligned} \tag{3.7.15}$$

and in the component form (we use the one-parametric $R$-matrix (3.7.2); the multiparametric case was considered in [74]), we have

$$\begin{gathered} T^{i_1}_{j_1}\, T^{i_2}_{j_2} - (-1)^{([i_1]+[j_1])([i_2]+[j_2])}\, T^{i_2}_{j_2}\, T^{i_1}_{j_1} = (q-q^{-1})\,(-1)^{([j_2][i_2]+[j_1]([i_2]+[j_2])}\, T^{i_1}_{j_2}\, T^{i_2}_{j_1} \\ (i_1 < i_2, \ \ j_1 < j_2), \\ T^{i_1}_{j_1}\, T^{i_2}_{j_2} = (-1)^{([i_1]+[j_1])([i_2]+[j_2])}\, T^{i_2}_{j_2}\, T^{i_1}_{j_1} \quad (i_1 < i_2, \ \ j_1 > j_2), \\ T^{i_1}_{j_1}\, T^{i_2}_{j_1} = (-1)^{[i_1][i_2]}\, q\, T^{i_2}_{j_1}\, T^{i_1}_{j_1} \ \ ([j_1] = 0, \ i_1 < i_2), \\ T^{i_1}_{j_1}\, T^{i_2}_{j_1} = (-1)^{([i_1]+1)([i_2]+1)}\, q^{-1}\, T^{i_2}_{j_1}\, T^{i_1}_{j_1} \ \ ([j_1] = 1, \ i_1 < i_2), \\ T^{i_1}_{j_1}\, T^{i_1}_{j_2} = (-1)^{[j_1][j_2]}\, q\, T^{i_1}_{j_2}\, T^{i_1}_{j_1} \ \ ([i_1] = 0, \ j_1 < j_2), \\ T^{i_1}_{j_1}\, T^{i_1}_{j_2} = (-1)^{([j_1]+1)([j_2]+1)}\, q^{-1}\, T^{i_1}_{j_2}\, T^{i_1}_{j_1} \ ([i_1] = 1, \ j_1 < j_2), \\ (T^{i_1}_{j_1})^2 = 0 \ \ ([i_1] \neq [j_1]). \end{gathered} \tag{3.7.16}$$

Relations (3.7.14)–(3.7.16) are the defining relations for the generators $T^i_j$ of the graded quantum algebra $\mathrm{Fun}(GL_q(N|M))$.

By using (3.3.1), the semiclassical analog of (3.7.15) can readily be deduced:

$$\big\{T_1,\ (-)^{(1)(2)}T_2(-)^{(1)(2)}\big\} = [T_1(-)^{(1)(2)}T_2(-)^{(1)(2)},\ r_{12}],$$

or in the component form, we have

$$(-1)^{[j_1]([i_2]+[j_2])}\ \big\{T^{i_1}_{j_1},\ T^{i_2}_{j_2}\big\} =$$

$$= T^{i_1}_{k_1}\,(-1)^{[k_1][i_2]}\,T^{i_2}_{k_2}\,(-1)^{[k_1][k_2]}\,r^{k_1k_2}_{j_1j_2} - r^{i_1i_2}_{k_1k_2}\,T^{k_1}_{j_1}\,(-1)^{[j_1][k_2]}\,T^{k_2}_{j_2}\,(-1)^{[j_1][j_2]},$$

where $\{.,.\}$ denotes the Poisson superbrackets

$$\{T^{i_1}_{j_1},\ T^{i_2}_{j_2}\} = -(-1)^{([i_1]+[j_1])([i_2]+[j_2])}\,\{T^{i_2}_{j_2},\ T^{i_1}_{j_1}\}.$$

The matrix $||T^i_j||$ is represented in the block form

$$T^i_j = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{3.7.17}$$

where the elements of the $N\times N$ matrix $||A^r_s||$ and of the $M\times M$ matrix $||D^\alpha_\beta||$ form the algebras $\mathrm{Fun}(GL_q(N))$ and $\mathrm{Fun}(GL_{q^{-1}}(M))$, respectively. Indeed, from (3.7.16) we have

$$\hat{R}^{i_1i_2}_{k_1k_2}[q]A^{k_1}_{\ j_1}\,A^{k_2}_{\ j_2} = A^{i_1}_{\ k_1}\,A^{i_2}_{\ k_2}\hat{R}^{k_1k_2}_{j_1j_2}[q],\quad \hat{R}^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}[q^{-1}]D^{\gamma_1}_{\ \beta_1}\,D^{\gamma_2}_{\ \beta_2} = D^{\alpha_1}_{\ \gamma_1}\,D^{\alpha_2}_{\ \gamma_2}\hat{R}^{\gamma_1\gamma_2}_{\beta_1\beta_2}[q^{-1}], \tag{3.7.18}$$

where $\hat{R}^{i_1i_2}_{k_1k_2}[q]$ and $\hat{R}^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}[q^{-1}]$ are standard $\mathrm{Fun}(GL_q(N))$ and $\mathrm{Fun}(GL_{q^{-1}}(M))$ $R$-matrices defined in (3.4.10). We assume that the quantum matrices $||A^r_s||$ and $||D^\alpha_\beta||$ are invertible. It means that the algebra $\mathrm{Fun}(GL_q(N|M))$ should be extended by the elements $\det_q^{-1}(A)$ and $\det_{q^{-1}}^{-1}(D)$ (see (3.4.38) and Definition **10**). In this case, from (3.7.18) and (3.4.12) we obtain

$$\hat{R}^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}[q](D^{-1})^{\gamma_1}_{\ \beta_1}\,(D^{-1})^{\gamma_2}_{\ \beta_2} = (D^{-1})^{\alpha_1}_{\ \gamma_1}\,(D^{-1})^{\alpha_2}_{\ \gamma_2}\hat{R}^{\gamma_1\gamma_2}_{\beta_1\beta_2}[q],$$
$$\hat{R}^{i_1i_2}_{k_1k_2}[q^{-1}](A^{-1})^{k_1}_{\ j_1}\,(A^{-1})^{k_2}_{\ j_2} = (A^{-1})^{i_1}_{\ k_1}\,(A^{-1})^{i_2}_{\ k_2}\hat{R}^{k_1k_2}_{j_1j_2}[q^{-1}].$$

For the elements of the rectangular matrices $||B^r_\beta||$ and $||C^\alpha_s||$ we obtain from (3.7.16) the commutation relations

$$\hat{R}^{i_1i_2}_{k_1k_2}[q]B^{k_1}_{\ \alpha_1}\,B^{k_2}_{\ \alpha_2} = -B^{i_1}_{\ \beta_1}\,B^{i_2}_{\ \beta_2}\hat{R}^{\beta_1\beta_2}_{\alpha_1\alpha_2}[q^{-1}],\quad B^{i_1}_{\ \alpha_1}C^{\alpha_2}_{\ i_2} = -C^{\alpha_2}_{\ i_2}B^{i_1}_{\ \alpha_1},$$
$$-\hat{R}^{\alpha_1\alpha_2}_{\beta_1\beta_2}[q^{-1}]C^{\beta_1}_{\ j_1}\,C^{\beta_2}_{\ j_2} = C^{\alpha_1}_{\ i_1}C^{\alpha_2}_{\ i_2}\hat{R}^{i_1i_2}_{j_1j_2}[q],$$
$$\hat{R}^{\alpha_1\alpha_2}_{\beta_1\beta_2}[q]C^{\beta_2}_{\ j_2}\,D^{\beta_1}_{\ \gamma_1} = P^{\alpha_1\alpha_2}_{\beta_1\beta_2}D^{\beta_1}_{\ \gamma_1}C^{\beta_2}_{\ j_2},\quad B^{i_2}_{\ \beta_2}\,D^{\alpha_1}_{\ \beta_1}P^{\beta_1\beta_2}_{\gamma_1\gamma_2} = D^{\alpha_1}_{\ \beta_1}B^{i_2}_{\ \beta_2}\hat{R}^{\beta_1\beta_2}_{\gamma_1\gamma_2}[q^{-1}],$$
$$A^{k_2}_{\ i_2}\,C^{\alpha_1}_{\ i_1}\hat{R}^{i_1i_2}_{j_1j_2}[q^{-1}] = C^{\alpha_1}_{\ i_1}A^{k_2}_{\ i_2}P^{i_1i_2}_{j_1j_2},\quad \hat{R}^{i_1i_2}_{j_1j_2}[q^{-1}]A^{j_2}_{\ k_2}B^{j_1}_{\ \beta_1} = P^{i_1i_2}_{j_1j_2}B^{j_1}_{\ \beta_1}A^{j_2}_{\ k_2},$$
$$A^i_{\ j}D^\alpha_\beta - D^\alpha_\beta A^i_{\ j} = (q-q^{-1})C^\alpha_{\ j}B^i_{\ \beta}.$$

By using these relations, one can prove that the elements of the matrix $X = (A - BD^{-1}C)$ satisfy the $RTT$ commutation relations

$$\hat{R}^{i_1i_2}_{k_1k_2}[q]X^{k_1}_{\ j_1}\,X^{k_2}_{\ j_2} = X^{i_1}_{\ k_1}\,X^{i_2}_{\ k_2}\hat{R}^{k_1k_2}_{j_1j_2}[q]. \tag{3.7.19}$$

It means that elements $X^i_j$ $(i,j = 1,\dots,N)$ generate a subalgebra $\mathrm{Fun}(GL_q(N))$ in $\mathrm{Fun}(GL_q(N|M))$. Assume that the quantum matrix $X = A - BD^{-1}C$ is also invertible. Then the same is valid for the matrix $||T^i_j||$, as it follows from the Gauss decomposition:

$$T = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} 1 & BD^{-1} \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} X & 0 \\ \hline 0 & D \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline D^{-1}C & 1 \end{array}\right), \tag{3.7.20}$$

and we have

$$T^{-1} = \left( \begin{array}{c|c} X^{-1} & -X^{-1}BD^{-1} \\ \hline -D^{-1}C\,X^{-1} & Y \end{array} \right) = \left( \begin{array}{c|c} X^{-1} & -A^{-1}BY \\ \hline -Y\,CA^{-1} & Y \end{array} \right), \tag{3.7.21}$$

where $Y = D^{-1}CX^{-1}BD^{-1} + D^{-1} = (D - CA^{-1}B)^{-1}$. By inverting relations (3.7.15), we obtain that elements of the inverse matrix $T^{-1}$ satisfy

$$\hat{R}_{12}\,(-)^{(1)(2)}T_2^{-1}(-)^{(1)(2)}T_1^{-1} = (-)^{(1)(2)}T_2^{-1}(-)^{(1)(2)}T_1^{-1}\,\hat{R}_{12}. \tag{3.7.22}$$

Now we note that Eq. (3.7.19) is simply obtained from (3.7.18) and (3.7.22) if we take into account (3.7.9) and (3.7.21).

In view of the existing of the inverse element (3.7.21), the algebra $\mathrm{Fun}(GL_q(N|M))$ with defining relations (3.7.15) is a Hopf algebra with usual structure mappings (3.2.3):

$$\Delta(T^i_k) = T^i_j \underline{\otimes}\, T^j_k, \;\; \epsilon(T^i_j) = \delta^i_j, \;\; S(T^i_j) = (T^{-1})^i_j,$$

where the product $\underline{\otimes}$ in the definition of $\Delta$ is understood as the graded direct product.

We define dual quantum multidimensional superplanes as a $\mathrm{Fun}(GL_q(N|M))$-comodule algebra $\mathcal{V}^*$ with generators $y_i$ ($i = 1, 2, \dots, N + M$) and left coaction (cf. (3.4.3)):

$$y_i \to \delta_T(y_i) = (1 \underline{\otimes}\, y_j)\,((T^{-1})^j_i \,\underline{\otimes}\, 1) \equiv (-1)^{([i]+[j])[j]}\,((T^{-1})^j_i \,\underline{\otimes}\, y_j). \tag{3.7.23}$$

This coaction is such that the pairing

$$Q = (y_i\, x^i) \tag{3.7.24}$$

is a co-invariant element $\delta_T(Q) = 1 \underline{\otimes}\, Q$ if the generators $x^i$ of the algebras $\mathcal{V}_\pm$ (3.7.12) are transformed according to (3.4.3). Assume that the grading of the coordinate $y_i$ is opposite to the grading of $x^i$, i.e., $[y_i] = [i] + 1$. Then the dual algebras $\mathcal{V}^*$, which are covariant under the transformations (3.7.23), have the following defining relations (cf. (3.7.12)):

$$\mathcal{V}^*_- : \;\; y_{\langle 2} y_{\langle 1} \left(\hat{R}'_{12} - q\right) = 0, \quad \mathcal{V}^*_+ : \;\; y_{\langle 2} y_{\langle 1} \left(\hat{R}'_{12} + q^{-1}\right) = 0, \tag{3.7.25}$$

where we have used new Hecke-type Yang–Baxter $R$-matrix: $\hat{R}'_{12} = (-)^{(1)}\hat{R}_{12}(-)^{(1)}$. We check directly the covariance of relations (3.7.25) under coaction (3.7.23):

$$\begin{aligned} y_{\langle 2} y_{\langle 1} \left(\hat{R}'_{12} \pm q^{\mp 1}\right) &\to y_{\langle 2} T_2^{-1}\, y_{\langle 1}\, T_1^{-1} \left(\hat{R}'_{12} \pm q^{\mp 1}\right) = \\ &= y_{\langle 2}\, y_{\langle 1} (-)^{((1)+1)(2)} T_2^{-1} (-)^{((1)+1)(2)} T_1^{-1} \left(\hat{R}'_{12} \pm q^{\mp 1}\right) = \\ &= y_{\langle 2}\, y_{\langle 1} (-)^{(1)(2)+(2)} T_2^{-1} (-)^{(1)(2)} T_1^{-1} \left(\hat{R}_{12} \pm q^{\mp 1}\right) (-)^{(2)} = \\ &= y_{\langle 2}\, y_{\langle 1} (-)^{(1)(2)+(2)} \left(\hat{R}_{12} \pm q^{\mp 1}\right) T_2^{-1} (-)^{(1)(2)} T_1^{-1}\, (-)^{(2)} = \\ &= y_{\langle 2}\, y_{\langle 1} \left((-)^{(1)}\hat{R}_{12}(-)^{(1)} \pm q^{\mp 1}\right) (-)^{((1)+1)(2)} T_2^{-1} (-)^{((1)+1)(2)} T_1^{-1}. \end{aligned}$$

Here we have used concise notation $(1 \underline{\otimes}\, y_j)\,((T^{-1})^j_i \,\underline{\otimes}\, 1) \equiv y_j (T^{-1})^j_i$. In the component form, Eqs. (3.7.25) are

$$\begin{aligned} \mathcal{V}^*_- : &\;\; q\, y_i\, y_j = -(-1)^{([i]+1)([j]+1)} y_j\, y_i \;\; (i < j), \quad (y_i)^2 = 0 \;\text{ if }\; [i] = 1, \\ \mathcal{V}^*_+ : &\;\; y_i\, y_j = (-1)^{([i]+1)([j]+1)} q\, y_j\, y_i \;\; (i < j), \quad (y_i)^2 = 0 \;\text{ if }\; [i] = 0. \end{aligned}$$

Covariant (with respect to coactions (3.4.3), (3.7.23)) cross-commutation relations for generators $x^i \in \mathcal{V}_\pm$ and $y_j \in \mathcal{V}^*_\pm$ are

$$x^{2\rangle}\, y_{\langle 2} = (-)^{(2)}\, y_{\langle 1}\, \hat{R}_{12}\, x^{1\rangle}.$$

Using these relations, we define covariant algebras $\mathcal{V}_\pm \,\sharp\, \mathcal{V}^*_\pm$ and $\mathcal{V}_\pm \,\sharp\, \mathcal{V}^*_\mp$ which are the cross-products of algebras $\mathcal{V}_\pm$ and $\mathcal{V}^*_\pm$. For $q^2 \neq -1$ one can easily check that the element $Q \in \mathcal{V}_\mp \,\sharp\, \mathcal{V}^*_\pm$ (defined in (3.7.24) and having the grading $[Q] = 1$) satisfies $Q^2 = 0$. Let $d$: $\mathcal{V}_- \,\sharp\, \mathcal{V}^*_+ \to \mathcal{V}_- \,\sharp\, \mathcal{V}^*_+$ be the linear map $d(f) = f \cdot Q$, where $f \in \mathcal{V}_- \,\sharp\, \mathcal{V}^*_+$, and we have $d^2(f) = 0$. Put $H(\mathcal{V}_- \,\sharp\, \mathcal{V}^*_+) = \mathrm{Ker}(d)/\mathrm{Im}(d)$. The map $d$ defines the structure of the Koszul complex on $\mathcal{V}_- \,\sharp\, \mathcal{V}^*_+$.

**Proposition 3.13** *(see [74]). $H(\mathcal{V}_- \,\sharp\, \mathcal{V}^*_+)$ is a one-dimensional subspace generated by*

$$\prod_{[i]=0} y_i \prod_{[j]=1} x^j \quad \mathrm{mod}\; (\mathrm{Im}(d)), \tag{3.7.26}$$

*and*

$$\delta_T \left( \prod_{[i]=0} y_i \prod_{[j]=1} x^j \right) = \mathrm{sdet}_q^{-1}(T) \,\underline{\otimes} \prod_{[i]=0} y_i \prod_{[j]=1} x^j \quad \mathrm{mod}\; (\mathrm{Im}(d)), \tag{3.7.27}$$

*where $\Delta(\mathrm{sdet}_q(T)) = \mathrm{sdet}_q(T) \,\underline{\otimes}\, \mathrm{sdet}_q(T)$. The element $\mathrm{sdet}_q(T)$ is called the quantum Berezinian (or quantum superdeterminant).*

We now compare the relations (3.7.15) with the graded Yang–Baxter equation (3.7.7). From this comparison we readily see that the finite-dimensional matrix representations for the generators $T^i_j$ of the quantum algebra $\mathrm{Fun}(GL_q(N|M))$ (the superanalogs of the representations (3.2.18)) can be chosen in the form

$$(T_1)_3 = (-)^{(1)(3)} R_{13} (-)^{(1)(3)} \equiv R^{(+)}_{13}, \;\; (T_1)_3 = (R^{-1})_{31} \equiv R^{(-)}_{13}. \tag{3.7.28}$$

From this, in an obvious manner, we obtain definitions of the quantum superalgebras which are dual to the algebras $\mathrm{Fun}(GL_q(N|M))$ (cf. Eqs. (3.2.19)):

$$\langle L^+_2, T_1 \rangle = (-)^{(1)(2)} R_{12} (-)^{(1)(2)} = R_{12}, \;\; \langle L^-_2, T_1 \rangle = R^{-1}_{21}, \tag{3.7.29}$$

where operator-valued matrices $L^\pm$ satisfy

$$\begin{aligned} \hat{R}_{12}\, L^\pm_2\, (-)^{(1)(2)}\, L^\pm_1\, (-)^{(1)(2)} &= L^\pm_2\, (-)^{(1)(2)}\, L^\pm_1\, (-)^{(1)(2)}\, \hat{R}_{12}, \\ \hat{R}_{12}\, L^+_2\, (-)^{(1)(2)}\, L^-_1\, (-)^{(1)(2)} &= L^-_2\, (-)^{(1)(2)}\, L^+_1\, (-)^{(1)(2)}\, \hat{R}_{12}. \end{aligned} \tag{3.7.30}$$

By using the identity $\hat{R}_{12}(-)^{(1)(2)} = (-)^{(1)(2)} \hat{R}_{12}$ (see (3.7.8)) for the $R$-matrix (3.7.1), one can deduce from (3.7.30) the standard reflection equation (3.2.31) for the matrix $L = S(L^-)L^+$.

Recall that the $R$-matrix (3.7.1) for $GL_q(N|M)$ is such that its diagonal blocks $\hat{R}^{i_1 i_2}_{k_1 k_2}[q]$ and $\hat{R}^{\alpha_1 \alpha_2}_{\gamma_1 \gamma_2}[q]$ are standard $\mathrm{Fun}(GL_q(N))$ and $\mathrm{Fun}(GL_q(M))$ $R$-matrices of the Hecke type and we have commutation relations (3.7.18), (3.7.19) for matrices $D$ and $X = A - BD^{-1}C$. Then one can write the quantum superdeterminant for $GL_q(N|M)$ by means of the definition (3.7.27) in the form [74, 155, 156]

$$\mathrm{sdet}_q^{-1}(T) = \det_{q^{-1}}(A - BD^{-1}C)^{-1} \det_{q^{-1}}(D), \tag{3.7.31}$$

where $\det_q$ is defined in (3.4.32). Then the algebra $\mathrm{Fun}(SL_q(N|M))$ is distinguished by the relation $\mathrm{sdet}_q(T) = 1$.

**Remark.** The standard formula for superdeterminant of supermatrix $T$ is deduced from the integral representation:

$$\mathrm{sdet}(T^M_N) \sim \int \prod_M (dE^M\, dF_M)\, \exp[iE^N\, (T^M_N)\, F_M], \qquad T^M_N = \begin{pmatrix} A^\xi_\eta & B^m_\eta \\ C^\xi_n & D^m_n \end{pmatrix}, \tag{3.7.32}$$

where supermatrix $T$ is given in the block form and we, respectively, divide the supervectors $E^N = (b^\eta,\, \beta^n)$ and $F_M = (c_\xi,\, \gamma_m)$ on even $\{\beta,\, \gamma\}$ and odd $\{b,\, c\}$ parts. Then we transform the quadratic expression $E^N\, (T^M_N)\, F_M$ to the "diagonal" form

$$E\,(T)\,F = b\,(A - B\,D^{-1}\,C)\,c + \widetilde{\beta}\,D\,\widetilde{\gamma},$$

by making the linear change of even variables $\beta = \widetilde{\beta} - b\,B\,D^{-1}$, $\gamma = \widetilde{\gamma} - D^{-1}\,C\,c$ . The Jacobian for such change of variables is equal to 1. After integration over $b, c$ and $\widetilde{\beta}, \widetilde{\gamma}$ in (3.7.32), we obtain

$$\mathrm{sdet}(T^M_N) = \det(A - B\,D^{-1}\,C)\,\frac{1}{\det D}. \tag{3.7.33}$$

The element $\mathrm{sdet}_q^{-1}(T)$, which appeared in Eqs. (3.7.27) and (3.7.31), is denoted as an inverse of the superdeterminant $\mathrm{sdet}_q(T)$, since the element $\mathrm{sdet}_q^{-1}(T)$ tends to $\mathrm{sdet}^{-1}(T)$ for $q \to 1$ in view of the standard formula (3.7.33). We also note that the meaning of Proposition **3.13** is to find the Jacobian of the supercoordinate transformation for the measure (3.7.26) of an integration over multidimensional quantum superplane.

The quantum supergroup $GL_q(N|M)$ was studied in detail from somewhat different positions in [156]. The simplest example of a quantum supergroup, $GL_q(1|1)$, has been investigated in many studies (see, for example, [141] and [157–160]). The $R$-matrices (3.6.5) can be used (see next Subsection **3.8**) to construct the supersymmetric Baxterized solutions of the Yang–Baxter equation (3.8.5) obtained in [161]. The Yangian limits of these solutions[16] were used to formulate integrable supersymmetric spin chains (see, e.g., [162]). The universal $R$-matrices for the linear quantum supergroups (and more generally for quantum deformations of finite-dimensional contragredient Lie (super)algebras) were constructed in [163].

### *3.8. $GL_q(N)$- and $GL_q(N|M)$-invariant Baxterized R-matrices. Dynamical R-matrices*

By Baxterization, we mean the construction of an $R$-matrix that depends not only on a deformation parameter $q$, but also on an additional complex spectral parameter $x$. We wish to find a solution $\hat{R}(x)$ of the Yang–Baxter equation with spectral parameter $x$ (see Eq. (3.8.2) below) satisfying the quantum invariance condition

$$T_1 T_2\, \hat{R}(x)\, (T_1 T_2)^{-1} = \hat{R}(x), \qquad \left(T^i_j \in \mathrm{Fun}(GL_q(N))\right).$$

Then we must seek it in the form [84]

$$\hat{R}(x) = b(x)(\mathbf{1} + a(x)\hat{R}) \tag{3.8.1}$$

(here $a(x)$ and $b(x)$ are certain functions of $x$), since by virtue of the Hecke relation (3.4.11), there exist only two basis matrices $\mathbf{1}$ and $\hat{R}$ that are invariants in the sense of the relations

[16] The corresponding $RTT$ algebra defines the Yangian of the Lie superalgebra $gl(n|m)$ [155].

$T_1T_2\ \hat{R}\ (T_1T_2)^{-1} = \hat{R}$ followed from (3.2.1). The Yang–Baxter equation with dependence on the spectral parameter is chosen in the form

$$\hat{R}_{12}(x)\,\hat{R}_{23}(xy)\,\hat{R}_{12}(y) = \hat{R}_{23}(y)\,\hat{R}_{12}(xy)\,\hat{R}_{23}(x). \tag{3.8.2}$$

Only the function $a(x)$ is fixed by this equation. Indeed, we substitute here (3.8.1) and take into account (3.1.11) and the Hecke condition (3.1.68). As a result, we obtain the equation [84]

$$a(x) + a(y) + \lambda a(x)a(y) = a(xy), \tag{3.8.3}$$

which is readily solved by the change of variables $a(x) = (1/\lambda)(\tilde{a}(x) - 1)$. After this, we obtain for the function $a$ the general solution

$$a(x) = (1/\lambda)(x^{\xi} - 1), \tag{3.8.4}$$

where for simplicity the arbitrary parameter $\xi$ can be set equal to $-2$. For convenience, we choose the normalizing function $b(x) = x$. Then the Baxterized $R$-matrix satisfying the Yang–Baxter equation (3.8.2) will have the form [113, 200, 237], [84]

$$\hat{R}(x) = b(x)\left(\mathbf{1} + (1/\lambda)(x^{-2} - 1)\hat{R}\right) = \frac{1}{\lambda}\,(x^{-1}\hat{R} - x\hat{R}^{-1}). \tag{3.8.5}$$

Remarkably, this matrix is written as the rational function of $\hat{R}$

$$\hat{R}(x) = \frac{(a^{-1}x - ax^{-1})}{\lambda\, x^2}\,\frac{\hat{R} - a\,x^2}{\hat{R} - a\,x^{-2}}, \qquad a = \mp q^{\pm 1}. \tag{3.8.6}$$

Below we call this $R$-matrix the *Hecke-type Baxterized R-matrix*. For the normalization adopted in (3.8.5) we obtain

$$\hat{R}(1) = 1, \quad \mathbf{P}^{\pm} = \frac{1}{[2]_q}\,\hat{R}(q^{\mp 1}), \tag{3.8.7}$$

and the unitarity condition holds[17]

$$\hat{R}(x)\,\hat{R}(x^{-1}) = \left(1 - \frac{(x - x^{-1})^2}{\lambda^2}\right). \tag{3.8.8}$$

This unitarity follows from rational representation (3.8.6) and can be readily deduced from the spectral decomposition

$$\hat{R}(x) = \frac{(x^{-1}q - xq^{-1})}{\lambda}\mathbf{P}^{+} + \frac{(xq - (xq)^{-1})}{\lambda}\mathbf{P}^{-},$$

where projectors $\mathbf{P}^{\pm}$ were defined in (3.4.21), (3.4.22). Note that we have obtained the Baxterized solution (3.8.5) of the Yang–Baxter equation (3.8.2) only using the braiding relations (3.1.6) and the Hecke condition (3.4.11) for the constant matrix $\hat{R}$. Thus, any constant Hecke solution of (3.1.6) (e.g., the multiparametric solution (3.6.2)) can be used for the construction of the Baxterized $R$-matrices (3.8.5).

[17] Strictly speaking, we have to renormalize the $R$-matrix (3.8.5): $\hat{R}(x) \to \lambda(x^{-1}q - xq^{-1})^{-1}\hat{R}(x)$, to obtain the unitarity condition with the unit matrix in the right-hand side of (3.8.8).

For the Baxterized $R$-matrix (3.8.5), constructed via skew-invertible Hecke-type $R$-matrix, one can deduce the cross-unitarity conditions

$$\mathrm{Tr}_{D(2)}\big(\hat{R}_1(x)P_{01}\hat{R}_1(z)\big) = \eta(x,z)\,D_0\,I_1, \qquad \mathrm{Tr}_{Q(1)}\big(\hat{R}_1(x)P_{23}\hat{R}_1(z)\big) = \eta(x,z)\,Q_3\,I_2, \tag{3.8.9}$$

where

$$\eta(x,z) = \frac{(x-x^{-1})(z-z^{-1})}{\lambda^2},$$

$P_{01}, P_{23}$ are permutations, matrices $D, Q$ were defined in (3.1.20) and spectral parameters $x, z$ are constrained by the condition

$$(x\,z)^2 = \frac{1}{1-\lambda \mathrm{Tr}(D)} =: b^2.$$

We stress that for the $GL(N|M)$-type $R$-matrix we have $b^2 = q^{2(N-M)}$.

Let $\hat{\Psi}_{12}$ be a skew-inverse matrix (3.1.18) for the Hecke $R$-matrix (3.1.68). Then, for the Baxterized $R$-matrix (3.8.5), one can define the skew-inverse Baxterized matrix $\hat{\Psi}(x)$:

$$\hat{\Psi}_{12}(x) = \frac{\lambda}{x^{-1}-x}\Big(\hat{\Psi}_{12} + \frac{\lambda}{b^{-2}-x^{-2}}\,D_1 Q_2\Big), \tag{3.8.10}$$

such that

$$\mathrm{Tr}_2\big(\hat{\Psi}_{12}(x)\,\hat{R}_{23}(x)\big) \;=\; P_{13} \;=\; \mathrm{Tr}_2\big(\hat{R}_{12}(x)\,\hat{\Psi}_{23}(x)\big). \tag{3.8.11}$$

Let $x_i$ and $p_j$ $(i,j = 1,\dots,N)$ be generators of the Heisenberg algebra:

$$[x_i,\, p_j] = i\,\hbar\,\delta_{ij} \qquad (i,j \leqslant N-1), \tag{3.8.12}$$

where $\hbar$ is a Planck constant. The dynamical Yang–Baxter equation is defined as follows [109, 233, 234] (see also [232, 235, 236]):

$$(Q_3^{-1}\,\hat{R}_{12}(p)\,Q_3)\,\hat{R}_{23}(p)\,(Q_3^{-1}\,\hat{R}_{12}(p)\,Q_3) = \hat{R}_{23}(p)\,(Q_3^{-1}\,\hat{R}_{12}(p)\,Q_3)\,\hat{R}_{23}(p), \tag{3.8.13}$$

where $Q := \mathrm{diag}(e^{ix_1}, e^{ix_2}, \dots, e^{ix_N})$. We seek the solution of (3.8.13) in the form (cf. (3.4.7))

$$\hat{R}_{12} = \hat{R}^{i_1 i_2}_{j_1 j_2}(p) = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\,a_{i_1 i_2}(p) + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\,b_{i_1 i_2}(p) \tag{3.8.14}$$

and require that this $R$-matrix satisfies the Hecke condition (3.1.68). Without limitation of generality one can put $b_{ii}(p) = 0$. Now we substitute (3.8.14) to the dynamical Yang–Baxter equation (3.8.13) and obtain the following constraints:

$$a_{ij}(p_1,\dots,p_N) = a_{ij}(p_i,p_j), \;\; b_{ij}(p_1,\dots,p_N) = b_{ij}(p_i,p_j), \tag{3.8.15}$$

and equations [109]

$$a_i^2 - \lambda a_i - 1 = 0, \quad b_{ij}(p_i,p_j) + b_{ji}(p_j,p_i) = \lambda, \qquad i \neq j, \tag{3.8.16}$$

$$a_{ij}(p_i,p_j)\,a_{ji}(p_j,p_i) - b_{ij}(p_i,p_j)\,b_{ji}(p_j,p_i) = 1, \qquad i \neq j, \tag{3.8.17}$$

$$b_{ij}\,b_{jk}\,b_{ki} + b_{ik}\,b_{kj}\,b_{ji} = 0, \qquad i \neq j \neq k \neq i, \tag{3.8.18}$$

$$b_{ij}(p_i+\hbar, p_j) = \frac{b_{ij}(p_i,p_j)\,a_i}{1/a_i + b_{ij}(p_i,p_j)}, \qquad b_{ij}(p_i, p_j+\hbar) = \frac{b_{ij}(p_i,p_j)/a_j}{a_j - b_{ij}(p_i,p_j)}, \tag{3.8.19}$$

where $a_i := a_{ii}$, Eqs. (3.8.16), (3.8.17) are consequences of the Hecke condition (3.1.68), while Eqs. (3.8.18), (3.8.19) follow from (3.8.13). The general solution of these equations for coefficients $b_{ij}(p)$ are [109]:

$$b_{ij}(p_i, p_j) = \frac{\lambda\, a_i^{p_i/\hbar}\, a_j^{-p_j/\hbar}\, b^0_{ij}}{a_i^{p_i/\hbar}\, a_j^{-p_j/\hbar}\, b^0_{ij} + a_i^{-p_i/\hbar}\, a_j^{p_j/\hbar}\, b^0_{ji}}, \tag{3.8.20}$$

where constants $b^0_{ij} := b_{ij}(0,0)$ have to obey the algebraic relations:

$$b^0_{ii} = 0, \quad b^0_{ij} + b^0_{ji} = \lambda, \quad b^0_{ij}\, b^0_{jk}\, b^0_{ki} + b^0_{ik}\, b^0_{kj}\, b^0_{ji} = 0. \tag{3.8.21}$$

The first equation in (3.8.16) has two solutions $a_i = \pm q^{\pm 1}$. Recall (see Subsection **3.7**) that if we take $a_i = q$, $\forall i$ (or $a_i = -q^{-1}$, $\forall i$), then we will have the case of the standard quantum group $GL_q(N)$ (or $GL_{-1/q}(N)$). But if we consider the mixing case, $a_i = q$ for $1 \leqslant i \leqslant K$ and $a_i = -q^{-1}$ for $K+1 \leqslant i \leqslant N$, then we come to the case of supergroups $GL_q(K|N-K)$. By considering the solution (3.8.20), it is clear that if $a_i = a_j$ (indices $i$ and $j$ 'have the same grading'), then $b_{ij}(p_i, p_j) = b_{ij}(p_i - p_j)$, but if $a_i = -1/a_j$ (the case of supergroups when indices $i$ and $j$ 'have the opposite grading'), then we deduce that $b_{ij}(p_i, p_j) = b_{ij}(p_i + p_j)$. Note that the only conditions on the parameters $a_{ij}(p)$ needed for fulfillment of the dynamical Yang–Baxter equation are listed in (3.8.17).

Now we demonstrate that every solution $R(p)$ given in (3.8.14), (3.8.17), (3.8.20) will lead to the solution $R(p,z)$ for the dynamical Yang–Baxter equation with spectral parameters

$$\hat{R}_{12}(p,\, y)\, Q_3\, \hat{R}_{23}(p,\, yz)\, Q_3^{-1}\, \hat{R}(p,\, z) = Q_3\, \hat{R}_{23}(p,\, z)\, Q_3^{-1}\, \hat{R}_{12}(p,\, yz)\, Q_3\, \hat{R}_{23}(p,\, y)\, Q_3^{-1}\,. \tag{3.8.22}$$

Indeed, it is not difficult to check, by using (3.8.13) and the Hecke relation for $\hat{R}(p)$, that the following matrices (cf. (3.8.5)):

$$\hat{R}(p,\, y) = y^{-1}\, \hat{R}(p) - y\, \hat{R}(p)^{-1}$$

are the solutions of (3.8.22). We note that these solutions satisfy the identity (cf. (3.8.8))

$$\hat{R}(p,\, y)\, \hat{R}(p,\, y^{-1}) = \left(\lambda^2 - (y - y^{-1})^2\right),$$

which is a kind of unitary condition for $\hat{R}(p)$ (if $y^* = y^{-1}$).

### *3.9. Quantum matrix algebras with spectral parameters. Yangians $Y_q(gl_N)$ and $Y(gl_N)$*

It is a remarkable fact that the relations (3.2.21), (3.2.22), with the Hecke $\hat{R}$-matrix, are written as follows:

$$\hat{R}_{12}(x)\, L_2(xy)\, L_1(y) = L_2(y)\, L_1(xy)\, \hat{R}_{12}(x), \tag{3.9.1}$$

$$L(x) := x^{-1} L^+ - x L^-, \tag{3.9.2}$$

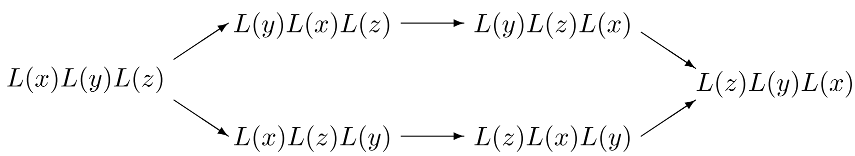
where $x$ and $y$ are arbitrary spectral parameters and $\hat{R}(x)$ is Baxterized $R$-matrix (3.8.5). Moreover, if we take the pairing of the relation (3.9.1) with the representation matrix $T^i_j$ acting in the third space and use (3.2.19), we obtain the Yang–Baxter equation (3.8.2) for the solution (3.8.5). Thus, in a certain sense, Eq. (3.9.1) generalizes (3.8.2).

Now we take the $GL_q(N)$-type Baxterized $R$-matrix (3.8.5) and consider Eqs. (3.9.1) as defining relations for new infinite-dimensional algebras with generators $(L_{(r)})^i_j$, which appeared in the expansion

$$L^i_j(x) = \sum_{r \geqslant 0} (L_{(r)})^i_j\, x^{-2r}. \tag{3.9.3}$$

This algebra is called *quantum* Yangian $Y_q(gl_N)$ and it is a subalgebra in a quantum affine algebra $U_q(\hat{gl}_N)$ (the *RTT* definition of $U_q(\hat{gl}_N)$ is given in [140], [112]). Formula (3.9.2) defines a homomorphism $Y_q(gl_N) \to U_q(gl_N)$ which is called *evaluation homomorphism.*

Let $R(x)$ be some solution of the Yang–Baxter equation (3.8.2) and there is an algebra with defining relations (3.9.1). It is known that (3.8.2) are associativity conditions of the unique ordering of monomials of the third degree $L_1(x)L_2(y)L_3(z)$ for the algebra (3.9.1). Indeed, we have the following diagram (the so-called "diamond" condition):




This diagram means that two different ways of reordering $L(x)L(y)L(z) \to L(z)L(y)L(x)$ (by means of (3.8.2)) give the same result.

Now we stress that, for the quantum algebra (3.9.1) with special $R$-matrix (3.8.5), defined by the Hecke $R$-matrix of the height $N$, the quantum determinant (the analog of (3.4.32)) can also be constructed [113]:

$$\det_q(L(x))\, \mathcal{E}_{\langle 12\dots N} = \mathcal{E}_{\langle 12\dots N}\, L_N(q^{N-1}x) \cdots L_2(qx)\, L_1(x) \quad \Leftrightarrow \tag{3.9.4}$$

$$\begin{aligned} \det_q(L(x)) &= \mathrm{Tr}_{1\dots N}\left(A_{1\to N}\, L_N(q^{N-1}x) \cdots L_2(qx)\, L_1(x)\right) = \\ &= \mathrm{Tr}_{1\dots N}\left(L_N(x)\, L_{N-1}(qx) \cdots L_1(q^{N-1}x)\, A_{1\to N}\right), \end{aligned} \tag{3.9.5}$$

$$\det_q(L(x))\, A_{1\to N} = L_N(x)\, L_{N-1}(qx) \cdots L_1(q^{N-1}x)\, A_{1\to N}, \tag{3.9.6}$$

where the rank-1 antisymmetrizer $A_{1\to N}$ has been introduced in (3.5.1). Equation (3.9.4) is self-consistent, since its right-hand side has the same symmetry as the left-hand side (the action on both sides of this equation by the projectors (3.8.7) $\mathbf{P}^+_k \sim \hat{R}_k(q^{-1})$ gives zero). The last form (3.9.5) of the quantum determinant $\det_q(L(x))$ is obtained with the help of (3.5.1) and (3.9.1).

**Proposition 3.14.** *The q-determinant $\det_q(L(x))$ is a generating function of central elements for the algebra (3.9.1) with the $GL_q(N)$-type Baxterized R-matrix (3.8.5).*

**Proof.** The centrality of $\det_q(L(x))$ means that $[L^i_j(xy), \det_q(L(x))] = 0 \; \forall x, y$. Indeed,

$$L_{N+1}(xy)\, \mathrm{Tr}_{1\dots N}\left(L_N(x)\, L_{N-1}(qx) \cdots L_1(q^{N-1}x)\, A_{1\to N}\right) = \tag{3.9.7}$$

$$\begin{aligned} = \mathrm{Tr}_{1\dots N}\Big(&\hat{R}^{-1}_N(y) \dots \hat{R}^{-1}_1(q^{1-N}y)\, L_{N+1}(x) \cdots L_2(q^{N-1}x) L_1(xy) \times \\ &\times \hat{R}_1(q^{1-N}y) \dots \hat{R}_N(y)\, A_{1\to N}\Big). \end{aligned} \tag{3.9.8}$$

Using the Yang–Baxter equation (3.8.2) and the representation of $A_{1\to N}$ in terms of the Baxterized elements (3.5.1), we deduce

$$\hat{R}_1(q^{1-N}y) \dots \hat{R}_N(y)\, A_{1\to N} = A_{2\to N+1}\, \hat{R}_1(y) \dots \hat{R}_N(q^{1-N}y).$$

By means of this relation and Eq. (3.9.6) one can rewrite (3.9.8) in the form

$$\begin{aligned} \det_q(L(x))\, \mathrm{Tr}_{1\dots N}\Big(&\hat{R}^{-1}_N(y) \dots \hat{R}^{-1}_1(q^{1-N}y)\, L_1(xy) A_{2\to N+1} \times \\ &\times \hat{R}_1(y) \dots \hat{R}_N(q^{1-N}y) A_{1\to N}\Big) = \end{aligned}$$

$$= \mathrm{det}_q(L(x))\, \big(N(y)^{-1}\, L(xy)\, N(y)\big)_{N+1} = \mathrm{det}_q(L(x))\, L_{N+1}(xy), \tag{3.9.9}$$

where matrices $N(y)$ and $N(y)^{-1}$ are defined by

$$(N(y))^{1\rangle}_{\langle N+1} = \mathcal{E}_{\langle 2\dots N+1}\, \hat{R}_1(y) \dots \hat{R}_N(q^{1-N}y)\, \mathcal{E}^{1\dots N\rangle},$$

$$(N(y)^{-1})^{N+1\rangle}_{\langle 1} = \mathcal{E}_{\langle 1\dots N}\, \hat{R}_N^{-1}(y) \dots \hat{R}_1^{-1}(q^{1-N}y)\, \mathcal{E}^{2\dots N+1\rangle}$$

and, for $GL(N)$-type Baxterized $R$-matrices, they are proportional to the unit matrix. Comparing (3.9.7) and (3.9.9), we obtain the statement of the Proposition. ■
We stress here that not for all Hecke-type Baxterized $R$-matrices the element $\mathrm{det}_q(L(x))$ is central for the algebra (3.9.1). The example is given by multiparametric Hecke $R$-matrices (3.6.3).

We now note that from the algebra (3.9.1), disregarding the particular representation (3.9.2) for the $L(x)$ operator, we can obtain a realization for the Yangian $Y(gl(N))$ [10, 143] (see also review papers [144, 145]). Indeed, in (3.8.2) and (3.9.1), we make the change of spectral parameters

$$x = \exp\left(-\frac{1}{2}\lambda(\theta - \theta')\right), \quad y = \exp\left(-\frac{1}{2}\lambda\theta'\right). \tag{3.9.10}$$

Then the relations (3.8.2) and (3.9.1) can be rewritten in the form

$$\hat{R}_{12}(\theta-\theta')\, \hat{R}_{23}(\theta)\, \hat{R}_{12}(\theta') = \hat{R}_{23}(\theta')\, \hat{R}_{12}(\theta)\, \hat{R}_{23}(\theta-\theta') \Rightarrow \tag{3.9.11}$$

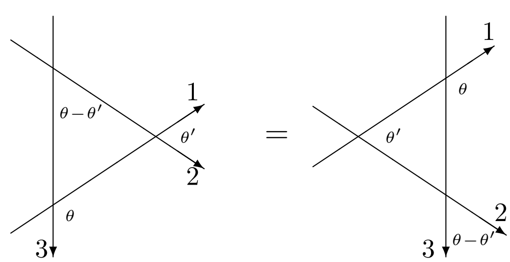
$$R_{23}(\theta-\theta')\, R_{13}(\theta)\, R_{12}(\theta') = R_{12}(\theta')\, R_{13}(\theta)\, R_{23}(\theta-\theta'), \tag{3.9.12}$$

$$\hat{R}_{12}(\theta-\theta')\, L_2(\theta)\, L_1(\theta') = L_2(\theta')\, L_1(\theta)\, \hat{R}_{12}(\theta-\theta'), \tag{3.9.13}$$

where we redefine $L$-operator $L(\theta) := L\left(\exp(-\frac{\lambda}{2}\theta)\right)$ and $R$-matrix

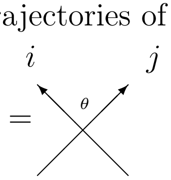
$$\hat{R}(\theta) := \hat{R}\left(e^{-\frac{\lambda}{2}\theta}\right) = \cosh(\lambda\theta/2) + \frac{1}{\lambda}\sinh(\lambda\theta/2)\,(\hat{R} + \hat{R}^{-1}). \tag{3.9.14}$$

Equations (3.9.12) have a beautiful graphical representation in the form of the triangle equation [4, 5]:




(3.9.15)

where the arrowed lines show trajectories of point particles, and the $R$-matrix

$R_{ij}(\theta) =$ 


describes a single act of the scattering of these particles. We now take the limit $\lambda = q - q^{-1} \to 0$ in Eq. (3.9.13). On the basis of (3.8.5), (3.9.14), we readily find that in this limit the matrix $\hat{R}(\theta)$ is equal to the Yang matrix:

$$\hat{R}(\theta) = (\mathbf{1} + \theta\, P_{12}) \quad \Rightarrow \quad R_{12}(\theta) = \theta\left(\mathbf{1} + \frac{P_{12}}{\theta}\right). \tag{3.9.16}$$

For the operators $L(\theta)$, we shall assume the expansion

$$L(\theta)^i_j = \sum_{k=0}^{\infty} T^{(k)}{}^i_j \theta^{-k}, \tag{3.9.17}$$

where $T^{(0)}{}^i_j = \delta^i_j$ and $T^{(k)}{}^i_j$ $(k > 0)$ are the generators of the Yangian $Y(gl(N))$ (see [10]). The defining relations for the Yangian $Y(gl(N))$ are obtained from (3.9.13) by substituting (3.9.16) and (3.9.17) (we give these relations in more general form of the super Yangian $Y(gl(N|M))$; see below (3.9.22)). The comultiplication for $Y(gl(N))$ obviously has the form

$$\Delta(L(\theta)^i_j) = L(\theta)^i_k \otimes L(\theta)^k_j. \tag{3.9.18}$$

The Yangian $Y(sl(N))$ is obtained from $Y(gl(N))$ after the imposition of an additional condition for the generators $T^{(k)}{}^i_j$:

$$\det{}_q(L(\theta)) = 1,$$

where the Yangian quantum determinant [146]

$$\det{}_q(L(\theta)) = \mathrm{Tr}_{1\dots N}\left(A^{\mathrm{cl}}_{1\to N}\, L_N(\theta - N + 1)\cdots L_2(\theta - 1)\, L_1(\theta)\right) \tag{3.9.19}$$

is obtained from (3.9.5) after substitution $q = e^h$, $x = \exp(-\frac{\lambda}{2}\theta) \sim e^{-h\theta}$ and taking the limit $h \to 0$ (or $\lambda \to 0$). In (3.9.19), we denote by $A^{\mathrm{cl}}_{1\to N}$ a classical antisymmetrizer:

$$A^{\mathrm{cl}}_{1\to N} = \lim_{q\to 1} A_{1\to N} = \frac{1}{N!}(1 + P_{N-1} + \ldots + P_1\cdots P_{N-1})\cdots(1 + P_2 + P_1P_2)(1 + P_1).$$

Since the $\hat{R}$-matrix (3.7.1), (3.7.2) (for the group $GL_q(N|M)$) satisfies the Hecke condition (3.4.11), the same Baxterized $R$-matrix (3.8.5) is appropriate for the supersymmetric case. Almost all statements of this subsection can be readily reformulated for the supersymmetric case. In particular, the Yangian $R$-matrix for $Y(gl(N|M))$ is deduced from (3.9.14) and has the form (cf. (3.9.16))

$$\hat{R}(\theta) = (\mathbf{1} + \theta\,\mathcal{P}_{12}), \tag{3.9.20}$$

where $\mathcal{P}_{12}$ is a supertransposition operator introduced in (3.7.3). The defining relations (3.9.1) should be modified for the super Yangian $Y(gl(N|M))$ (cf. (3.7.15)):

$$\hat{R}_{12}(\theta - \theta')\,(-)^{(1)(2)}\,L_2(\theta)\,(-)^{(1)(2)}\,L_1(\theta') = (-)^{(1)(2)}\,L_2(\theta')\,(-)^{(1)(2)}\,L_1(\theta)\,\hat{R}_{12}(\theta - \theta'), \tag{3.9.21}$$

while the form of the comultiplication (3.9.18) (where $\otimes$ is the graded tensor product) is unchanged. Taking into account (3.9.17) and (3.9.20), we obtain the component form of the defining relations (3.9.21) for $Y(gl(N|M))$:

$$[T^{(r)}{}^i_j,\, T^{(s+1)}{}^k_l\} - [T^{(r+1)}{}^i_j,\, T^{(s)}{}^k_l\} = (-1)^{[k][i]+[k][j]+[i][j]}\left(T^{(s)}{}^k_j\, T^{(r)}{}^i_l - T^{(r)}{}^k_j\, T^{(s)}{}^i_l\right), \tag{3.9.22}$$

where $r, s \geqslant 0$, $T^{(0)}{}^i_j = (-1)^{[i]}\,\delta^i_j$, the grading $[i] = 0, 1 \bmod(2)$ is defined in (3.7.4) and $[a, b\}$ denotes a supercommutator $[a, b\} := a\,b - (-1)^{[a][b]}b\,a$, $[a] = \deg(a)$.

The relations (3.9.13), (3.9.21) play an important role in the quantum inverse scattering method [7–9]. Equations (3.9.12) are the conditions of factorization of the $S$-matrices in certain exactly solvable two-dimensional models of quantum field theory (see [4, 5]). The matrix representations for the operators (3.9.2) satisfying (3.9.1) lead to the formulation of lattice integrable systems (see, for example, [103]). These questions will be discussed in more detail in the final section of the paper.

Another interesting presentations of the quantum operators $L(x)$, which satisfy (3.9.1), are given in [147, 148]. In paper [148], to construct $L$-operator, we use Eq. (3.9.1) with Baxterized $R$-matrix which is defined by means of the multiparametric $R$-matrix (3.6.3). These $L$-operators were applied to the formulation of 3-dimensional integrable models.

The super Yangians $Y(gl(N|M))$ and their representations have been discussed in [149, 155]. The quantum Berezinian for the Yangian (an analog of (3.7.31) and superanalog of (3.9.19)) was introduced in [155].

### *3.10. The quantum groups $SO_q(N)$ and $Sp_q(2n)$ (B, C, and D series)*

#### *3.10.1. Spectral decomposition for $SO_q(N)$- and $Sp_q(2n)$-type R-matrices*

In the remarkable paper [42], the quantum groups[18] $SO_q(N)$ and $Sp_q(N)|_{N=2n}$ were studied as Hopf algebras with the defining $RTT$ relations (3.2.1). These quantum groups are quantum deformations of Lie groups $SO(N)$ ($B_n$ and $D_n$ series, respectively, for $N = 2n+1$ and $N = 2n$) and $Sp(2n)$ ($C_n$ series). It was shown in [42] that $SO_q(N)$- and $Sp_q(2n)$-type $R$-matrices (used in the $RTT$ algebra (3.2.1)) have the form

$$R_{12} = \sum_{i,j} q^{(\delta_{ij}-\delta_{ij'})}\, e_{ii}\otimes e_{jj} + \lambda\sum_{i>j} e_{ij}\otimes e_{ji} - \lambda\sum_{i>j} q^{\rho_i-\rho_j}\,\epsilon_i\,\epsilon_j\, e_{ij}\otimes e_{i'j'}, \tag{3.10.1}$$

$$\hat{R}_{12} := P_{12}\,R_{12} = \sum_{i,j} q^{(\delta_{ij}-\delta_{ij'})}\, e_{ij}\otimes e_{ji} + \lambda\sum_{i<j} e_{ii}\otimes e_{jj} - \lambda\sum_{i>j} q^{\rho_i-\rho_j}\,\epsilon_i\,\epsilon_j\, e_{i'j}\otimes e_{ij'}, \tag{3.10.2}$$

where

$$\begin{gathered}\epsilon_i = +1\ \forall i \ \ (\text{for}\ \ SO_q(N)),\\ \epsilon_i = +1\ (1\leqslant i\leqslant n),\ \ \epsilon_i = -1\ (n+1\leqslant i\leqslant 2n)\ \ (\text{for}\ \ Sp_q(2n)),\\ (\rho_1,\dots,\rho_N) = \begin{cases} (n-\frac{1}{2},\, n-\frac{3}{2},\dots,\frac{1}{2},\, 0,\, -\frac{1}{2},\,\dots,-n+\frac{1}{2}), & B:(SO_q(2n+1)),\\ (n,\, n-1,\,\dots,1,\, -1,\dots,\, 1-n,\, -n), & C:(Sp_q(2n)),\\ (n-1,\, n-2,\dots,1,\, 0,\, 0,\, -1,\dots,1-n), & D:(SO_q(2n)).\end{cases}\end{gathered} \tag{3.10.3}$$

We deduce these $R$-matrices in Subsection **3.11.2** below. The matrices (3.10.1) satisfy not only the Yang–Baxter equation (3.1.2), (3.1.6), but also the cubic characteristic equation (3.1.72) (see Eq. (3.1.64) for $M = 3$):

$$(\hat{R}-q\mathbf{1})(\hat{R}+q^{-1}\mathbf{1})(\hat{R}-\nu\mathbf{1}) = 0, \tag{3.10.4}$$

where $\nu = \epsilon q^{\epsilon-N}$ is a "singlet" eigenvalue of $\hat{R}$, and the case $\epsilon = +1$ corresponds to the orthogonal groups $SO_q(N)$ ($B_n$ and $D_n$ series), while the case $\epsilon = -1$ corresponds to the symplectic groups $Sp_q(2n)$ ($C_n$ series). The projectors (3.1.66) arising from the characteristic equation (3.10.4) can be written as follows [42]:

$$\begin{gathered}\mathbf{P}^{\pm} = \frac{(\hat{R}\pm q^{\mp1}\mathbf{1})(\hat{R}-\nu\mathbf{1})}{(q+q^{-1})(q^{\pm1}\mp\nu)} \equiv \frac{1}{q+q^{-1}}\left(\pm\hat{R}+q^{\mp1}\mathbf{1}+\mu_{\pm}\mathbf{K}\right),\\ \mathbf{P}^{0} = \frac{(\hat{R}-q\mathbf{1})(\hat{R}+q^{-1}\mathbf{1})}{(\nu-q)(q^{-1}+\nu)} \equiv \mu^{-1}\mathbf{K},\end{gathered} \tag{3.10.5}$$

[18] We often use the short notation $SO_q(N)$ and $Sp_q(N)$ (for quantum groups) instead of more precise notation for algebras $\mathrm{Fun}(SO_q(N))$ and $\mathrm{Fun}(Sp_q(N))$.

where

$$\mu = \frac{(q-\nu)(q^{-1}+\nu)}{\lambda\nu} = \frac{\lambda + \nu^{-1} - \nu}{\lambda} = (1 + \epsilon[N-\epsilon]_q),$$

$$\mu_{\pm} = \pm \frac{\lambda}{(1 \mp q^{\pm 1}\nu^{-1})} = \mp \frac{\nu \pm q^{\mp 1}}{\mu}, \qquad \lambda := q - q^{-1}.$$

We also give the relations between the parameters $\nu$, $\mu$, $\mu_{\pm}$ that we introduced:

$$q\mu_+ - q^{-1}\mu_- = \nu(\mu_+ + \mu_-), \;\; \mu_+ + \mu_- = -\frac{q+q^{-1}}{\mu},$$

which are helpful in various calculations with projectors (3.10.5). For convenience, we define in (3.10.5) the operator $\mathbf{K}^{i_1 i_2}_{j_1 j_2}$, which projects $\hat{R}$ onto the "singlet" eigenvalue $\nu$:

$$\mathbf{K}\,\hat{R} = \hat{R}\,\mathbf{K} = \nu\,\mathbf{K}, \quad (\mathbf{K}^2 = \mu\mathbf{K}). \tag{3.10.6}$$

Then the characteristic equation (3.10.4) is written in another form (cf. (3.1.68))

$$\hat{R} - \hat{R}^{-1} - \lambda + \lambda\,\mathbf{K} = 0. \tag{3.10.7}$$

The spectral decomposition (3.1.67) for the $SO_q(N)$- and $Sp_q(2n)$-type $R$-matrices is

$$\hat{R} = q\,\mathbf{P}^+ - q^{-1}\,\mathbf{P}^- + \nu\,\mathbf{K}.$$

Note that in the semiclassical limit (3.3.1), when $q = e^h \to 1$, the characteristic equation (3.10.7) is reduced to the relation

$$\frac{1}{2}(r_{12} + r_{21}) = P_{12} - \epsilon\mathbf{K}^{(0)}_{12}, \tag{3.10.8}$$

where $(P_{12})^{i_1 i_2}_{j_1 j_2} \equiv (P)^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the permutation matrix, and in the right-hand side of (3.10.8), we obtain split Casimir operators for *so* and *sp* Lie algebras (see [176, 177] and references therein). Thus, as in the $GL_q(N)$ case (3.3.8), the semiclassical limit (3.10.8) of the characteristic equation fixes the ad-invariant part of the classical $r$-matrix. Here we have used an expansion of the matrix $\mathbf{K} = \mathbf{K}^{(0)} + h\mathbf{K}^{(1)} + O(h^2)$, where the first term is

$$(\mathbf{K}^{(0)})^{i_1 i_2}_{j_1 j_2} = (C_0)^{i_1 i_2}(C_0^{-1})_{j_1 j_2} \Rightarrow \mathbf{K}^{(0)}_{12} = C_0^{12\rangle}\,(C_0^{-1})_{\langle 12}\,. \tag{3.10.9}$$

The matrices $(C_0)^{ij}$: $(C_0)^2 = \epsilon$, $(C_0)^t = \epsilon C_0$ are the metric (symmetric) and symplectic (antisymmetric) matrices, respectively, for the groups $SO(N)$ and $Sp(2n)$. The semiclassical expansion for the projectors (3.10.5) and (3.10.47) has the form

$$\begin{gathered} \mathbf{P}^{\pm}_{\rm cl} = \tfrac{1}{2}\left((\mathbf{1} \pm P) \pm hP\tilde{r} - (1 \pm \epsilon)\mathbf{P}^0_{\rm cl}\right), \\ \mathbf{P}^0_{\rm cl} = \tfrac{\epsilon}{N}\left(\mathbf{K}^{(0)} + h\mathbf{K}^{(1)}\right), \end{gathered} \tag{3.10.10}$$

where the semiclassical matrix $\tilde{r}$ (3.3.8) (which satisfies the modified classical Yang–Baxter equation) is given by the formula

$$\tilde{r} = r_{12} - P_{12} + \epsilon K^{(0)}_{12} = -r_{21} + P_{12} - \epsilon K^{(0)}_{12}.$$

The ranks of the quantum projectors (3.10.5) are equal (for $q$ which is not the root of unity) to the ranks of the projectors (3.10.10), which are readily calculated in the classical limit $h = 0$. Accordingly, we have [42]:
1) for the groups $SO_q(N)$

$$\mathrm{rank}(P^{(+)}) = \frac{N(N+1)}{2} - 1, \; \mathrm{rank}(P^{(-)}) = \frac{N(N-1)}{2}, \; \mathrm{rank}(P^{(0)}) = 1 \; ; \tag{3.10.11}$$

2) for the groups $Sp_q(2n)$

$$\mathrm{rank}(P^{(+)}) = \frac{N(N+1)}{2}, \; \mathrm{rank}(P^{(-)}) = \frac{N(N-1)}{2} - 1, \; \mathrm{rank}(P^{(0)}) = 1 \; . \tag{3.10.12}$$

Since the rank of projector $P^{(0)}$ is equal to 1, we can write

$$(P^{(0)})^{i_1 i_2}_{j_1 j_2} = \alpha C^{i_1 i_2} \overline{C}_{\, j_1 j_2} \quad \Rightarrow \quad \mathbf{K}^{i_1 i_2}_{j_1 j_2} = C^{i_1 i_2} \overline{C}_{\, j_1 j_2}, \tag{3.10.13}$$

where in view of the second equation in (3.10.6), we have $\overline{C}_{\, i_1 i_2} C^{i_1 i_2} = \mu = \alpha^{-1}$.

*3.10.2. Quantum algebras* $\mathrm{Fun}(SO_q(N))$, $\mathrm{Fun}(Sp_q(2n))$ *and their dual algebras*

The number of generators $T^i_j$ $(i, j = 1, \dots, N)$ for the algebras $\mathrm{Fun}(SO_q(N))$ and $\mathrm{Fun}(Sp_q(2n))$ $(2n = N)$, which satisfy the $RTT$ relations (3.2.1)

$$R^{i_1 i_2}_{j_1 j_2} \, T^{j_1}_{k_1} \, T^{j_2}_{k_2} = T^{i_2}_{j_2} \, T^{i_1}_{j_1} \, R^{j_1 j_2}_{k_1 k_2}, \tag{3.10.14}$$

coincides with dimensions of the groups $SO(N)$ and $Sp(2n)$ in the undeformed case, since for $T^i_j$ the following subsidiary conditions are imposed:

$$T C T^t C^{-1} = C T^t C^{-1} T = 1 \Rightarrow \tag{3.10.15}$$

$$T_1 T_2 \, C^{12\rangle} = C^{12\rangle}, \;\; C^{-1}_{\langle 12} \, T_1 T_2 = C^{-1}_{\langle 12}. \tag{3.10.16}$$

These relations directly generalize the classical conditions for the elements of the groups $SO(N)$ and $Sp(2n)$. The matrices $C^{ij}$, $C^{-1}_{kl}$, which are understood in (3.10.16) as elements in $V_N \otimes V_N$ (1 and 2 label the spaces $V_N$), are the $q$-analogs of the metric and symplectic matrices $C_0$ for $SO(N)$ and $Sp(N)$, respectively. The explicit form of these matrices, which is given in [42] (see also Subsection **3.11**), is not important for us, but we stress that the following equation holds:

$$C^{-1} = \epsilon C, \tag{3.10.17}$$

where $\epsilon = +1$ and $\epsilon = -1$, respectively, for $SO_q(N)$ and $Sp_q(N)$ cases. Substituting the $R$-matrix representations (3.2.18) for $T^i_j$ in the relations (3.10.15), we obtain the following conditions on the $R$-matrices:

$$R_{12} = C_1 (R^{t_1}_{12})^{-1} C^{-1}_1 = C_2 (R^{-1}_{12})^{t_2} C^{-1}_2, \tag{3.10.18}$$

where, as usual, $C_1 = C \otimes I$ and $C_2 = I \otimes C$. As consequences of (3.10.18), we have the equation

$$R^{t_1 t_2}_{12} = C^{-1}_1 C^{-1}_2 R_{12} C_1 C_2 \tag{3.10.19}$$

and also subsidiary conditions

$$L^{\pm}_2 \, L^{\pm}_1 \, C^{12\rangle} = C^{12\rangle}, \;\; C^{-1}_{\langle 12} \, L^{\pm}_2 \, L^{\pm}_1 = C^{-1}_{\langle 12} \tag{3.10.20}$$

on the generators of the dual $U_q(so(N))$ and $U_q(sp(N=2n))$ algebras (3.2.21), (3.2.22):

$$\hat{R}_{12}\, L_2^{\pm}\, L_1^{\pm} = L_2^{\pm}\, L_1^{\pm}\, \hat{R}_{12}, \quad \hat{R}_{12}\, L_2^{+}\, L_1^{-} = L_2^{-}\, L_1^{+}\, \hat{R}_{12}. \tag{3.10.21}$$

The semiclassical analogs of the conditions (3.10.18) and (3.10.19) have the form

$$r_{12} = -(C_0)_1 r_{12}^{t_1}(C_0)_1^{-1} = -(C_0)_2 r_{12}^{t_2}(C_0)_2^{-1} = (C_0)_1(C_0)_2 r_{12}^{t_1 t_2}(C_0)_1^{-1}(C_0)_2^{-1}.$$

It follows from Eqs. (3.10.15) and (3.10.17) that the antipode $S(T) = C\,T^t\,C^{-1}$ for the Hopf algebras $\mathrm{Fun}(SO_q(N))$ and $\mathrm{Fun}(Sp_q(N))$ satisfies the relation

$$S^2(T) = (CC^t)T(CC^t)^{-1}, \tag{3.10.22}$$

which is analogous to (3.2.5). Thus, the matrix $D$ that defines the quantum trace for the quantum groups of the $SO$ and $Sp$ series can be chosen in the form

$$D = \epsilon\nu\, CC^t \quad \Leftrightarrow \quad D^i_j = \nu\, C^{ik}\, C^{-1}_{jk}, \tag{3.10.23}$$

where we take into account (3.10.17). Here we choose the numerical factor $\epsilon\nu$ in order to relate (3.10.23) with the general definitions of $D$-matrix (3.1.20), (3.1.22).

We now note that the matrix $C^{12\rangle}\, C^{-1}_{\langle 12} \in \mathrm{Mat}(N) \otimes \mathrm{Mat}(N)$ projects any vector $X^{12\rangle}$ onto the vector $C^{12\rangle}$, i.e., the rank of the projector $C^{12\rangle}C^{-1}_{\langle 12}$ is 1. In addition, from (3.10.16) we have

$$C^{12\rangle}C^{-1}_{\langle 12}\, T_1\, T_2 = T_1\, T_2\, C^{12\rangle}C^{-1}_{\langle 12},$$

which means that the projector $C^{12\rangle}C^{-1}_{\langle 12}$ should be a polynomial in $\hat{R}$. Therefore, $C^{12\rangle}C^{-1}_{\langle 12} \sim \mathbf{P}^0_{12}$, and, as it was established in [42] (cf. (3.10.13)),

$$C^{12\rangle}C^{-1}_{\langle 12} \equiv \mathbf{K}_{12}. \tag{3.10.24}$$

Using this relation, $RTT$ relations (3.10.14), and equations (3.10.7), one can deduce

$$T_1\, T_2\, \mathbf{K}_{12} = \mathbf{K}_{12}\, T_1\, T_2 = \tau(T)\, \mathbf{K}_{12}, \tag{3.10.25}$$

where we defined the scalar element $\tau = \mu^{-1} C^{-1}_{\langle 12}\, T_1 T_2\, C^{12\rangle}$. Comparing Eq. (3.10.25) with Eqs. (3.10.15) and (3.10.16), we conclude that $\tau = 1$. Therefore, for the correct definition of the quantum groups $SO_q(N)$ and $Sp_q(N)$ we should require the centrality of the element $\tau$ in the $RTT$ algebra (the centrality of the element $\tau$ is discussed below after Eq. (3.11.37)).

We note that Eqs. (3.10.6), (3.10.24) are equivalent to the relations

$$\hat{R}_{12}\, C^{12\rangle} = \nu\, C^{12\rangle}, \quad C^{-1}_{\langle 12}\, \hat{R}_{12} = \nu\, C^{-1}_{\langle 12}, \tag{3.10.26}$$

which give the possibility to rewrite conditions (3.10.20) for the generators $L^i_j = (S(L^-)L^+)^i_j$, $\bar{L}^i_j = (L^+ S(L^-))^i_j$ of the reflection equation algebras (3.2.31), (3.2.32) in the form

$$L_1\, \hat{R}_{12}\, L_1\, C^{12\rangle} = \nu\, C^{12\rangle}, \quad C^{-1}_{\langle 12}\, L_1\, \hat{R}_{12}\, L_1 = \nu\, C^{-1}_{\langle 12},$$

$$\overline{L}_2\, \hat{R}_{12}\, \overline{L}_2\, C^{12\rangle} = \nu\, C^{12\rangle}, \quad C^{-1}_{\langle 12}\, \overline{L}_2\, \hat{R}_{12}\, \overline{L}_2 = \nu\, C^{-1}_{\langle 12}.$$

By making use of the statements of Proposition **3.6**, we construct the central elements (3.2.29) for the algebras $U_q(so(N))$ and $U_q(sp(N=2n))$ as

$$C_{(M)} = \mathrm{Tr}_D \left( L^M \right) \equiv \mathrm{Tr} \left( D\, L^M \right), \tag{3.10.27}$$

where the quantum trace matrix $D$ is defined in (3.10.23). The elements (3.10.27) are quantum analogs of the Casimir operators for the algebras $U_q(so(N))$ and $U_q(sp(N=2n))$.

We now present some important relations for the matrices $\hat{R}$ and $\mathbf{K}$; many of them are given, in the same form or other, in [42]. We note first that in accordance with (3.1.15) we have

$$\mathbf{K}_{12}\, \hat{R}_{23}\, \hat{R}_{12} = \hat{R}_{23}\, \hat{R}_{12}\, \mathbf{K}_{23} \Leftrightarrow \hat{R}_{12}\, \hat{R}_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, \hat{R}_{12}\, \hat{R}_{23}. \tag{3.10.28}$$

Further, from Eqs. (3.10.18) and (3.10.24) (or substituting the matrix representations (3.2.18) in (3.10.25) for $\tau = 1$), we obtain

$$\begin{gathered} \hat{R}_{12}^{\pm 1}\, \hat{R}_{23}^{\pm 1}\, \mathbf{K}_{12} = P_{12}\, P_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, P_{12}\, P_{23}, \\ \mathbf{K}_{12}\, \hat{R}_{23}^{\pm 1}\, \hat{R}_{12}^{\pm 1} = \mathbf{K}_{12}\, P_{23}\, P_{12} = P_{23}\, P_{12}\, \mathbf{K}_{23}. \end{gathered} \tag{3.10.29}$$

A consequence of these relations is the equations

$$\begin{gathered} \hat{R}_{23}^{\pm 1}\, \mathbf{K}_{12}\, \hat{R}_{23}^{\pm 1} = \hat{R}_{12}^{\mp 1}\, \mathbf{K}_{23}\, \hat{R}_{12}^{\mp 1} \Leftrightarrow \hat{R}_{12}\, \hat{R}_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, \hat{R}_{12}^{-1}\, \hat{R}_{23}^{-1}, \\ \hat{R}_{23}\, \hat{R}_{12}\, \mathbf{K}_{23} = \mathbf{K}_{12}\, \hat{R}_{23}^{-1}\, \hat{R}_{12}^{-1}. \end{gathered} \tag{3.10.30}$$

In particular, taking into account the characteristic equation (3.10.7), we obtain the identity

$$(\hat{R}_{12} - \lambda)\, \mathbf{K}_{23}\, (\hat{R}_{12} - \lambda) = (\hat{R}_{23} - \lambda)\, \mathbf{K}_{12}\, (\hat{R}_{23} - \lambda)$$

or

$$\begin{gathered} \hat{R}_{12}\, \mathbf{K}_{23} \hat{R}_{12} = \hat{R}_{23}^{-1}\, \mathbf{K}_{12}\, \hat{R}_{23}^{-1} = \\ = \hat{R}_{23}\, \mathbf{K}_{12}\, \hat{R}_{23} + \lambda(\hat{R}_{12}\, \mathbf{K}_{23} - \mathbf{K}_{12}\, \hat{R}_{23} - \hat{R}_{23}\, \mathbf{K}_{12} + \mathbf{K}_{23}\, \hat{R}_{12}) + \lambda^2(\mathbf{K}_{12} - \mathbf{K}_{23}), \end{gathered} \tag{3.10.31}$$

which will be used in Subsection **3.12**. Equation (3.10.24) leads to the identities

$$\mathbf{K}_{12}\, \mathbf{K}_{23} = \mathbf{K}_{12}\, P_{23}\, P_{12} = P_{23}\, P_{12}\, \mathbf{K}_{23}, \;\; \mathbf{K}_{23}\, \mathbf{K}_{12} = P_{12}\, P_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, P_{12}\, P_{23}, \tag{3.10.32}$$

from which we immediately obtain

$$\mathbf{K}_{12}\, \mathbf{K}_{23}\, \mathbf{K}_{12} = \mathbf{K}_{12}, \;\; \mathbf{K}_{23}\, \mathbf{K}_{12}\, \mathbf{K}_{23} = \mathbf{K}_{23}. \tag{3.10.33}$$

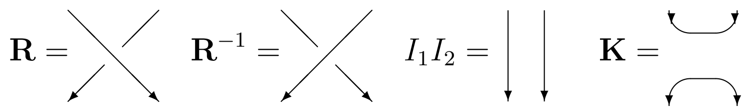
We now compare the relations (3.10.29) and (3.10.32). The result of this comparison is the equations

$$\begin{gathered} \hat{R}_{23}^{\pm 1}\, \hat{R}_{12}^{\pm 1}\, \mathbf{K}_{23} = \mathbf{K}_{12}\, \mathbf{K}_{23} = \mathbf{K}_{12}\, \hat{R}_{23}^{\pm 1}\, \hat{R}_{12}^{\pm 1}, \\ \hat{R}_{12}^{\pm 1}\, \hat{R}_{23}^{\pm 1}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, \hat{R}_{12}^{\pm 1}\, \hat{R}_{23}^{\pm 1}. \end{gathered} \tag{3.10.34}$$

We now apply to the first of the chain of equations in (3.10.34) the matrix $\mathbf{K}_{12}$ from the right (or $\mathbf{K}_{23}$ from the left) and take into account (3.10.6) and (3.10.33). We then obtain

$$\mathbf{K}_{23}\, \hat{R}_{12}^{\pm 1}\, \mathbf{K}_{23} = \nu^{\mp 1} \mathbf{K}_{23}, \;\; \mathbf{K}_{12}\, \hat{R}_{23}^{\pm 1}\, \mathbf{K}_{12} = \nu^{\mp 1} \mathbf{K}_{12}. \tag{3.10.35}$$

The braid relation (3.1.11) and Eqs. (3.10.6), (3.10.7), (3.10.35) define the $R$-matrix representation of the Birman–Murakami–Wenzl algebra [229] (see also Subsection **4.4** below). Equations (3.10.28), (3.10.30), (3.10.31), (3.10.33), and (3.10.34) directly follow from this definition. As we shall see in Subsection **3.12**, the relations for the Birman–Murakami–Wenzl algebra will be sufficient for the construction of $SO_q(N)$ and $Sp_q(2n)$-symmetric Baxterized $\hat{R}(x)$-matrices. The relations (3.10.28), (3.10.30), and (3.10.33)–(3.10.35) have a natural graphical representation in the form of relations for braids and links if we use the diagrammatic technique (only 3 of these operators are independent in view of (3.10.7)):




(3.10.36)

*3.10.3. Quantum traces and quantum hyperplanes for $SO_q(N)$ and $Sp_q(N)$*

We now give some important relations for the quantum trace (3.2.12) corresponding to the quantum groups $SO_q(N)$ and $Sp_q(N)$. Similar relations for the $q$-trace (3.2.14) can be derived exactly in the same way. From the definitions of the matrix $\mathbf{K}$ (3.10.24) and the matrix $D$ (3.10.23), we obtain

$$\mathrm{Tr}_{q2}(\mathbf{K}_{12}) = \nu\, I_{(1)}. \tag{3.10.37}$$

We use the relations (3.10.19) and the definition of the quantum trace (3.2.12) with the matrix $D$ (3.10.23); then, for an arbitrary quantum matrix $E^i_j$, we obtain the relations

$$\begin{gathered} \nu\, \hat{R}^n_{12}\, E_1\, \mathbf{K}_{12} = \mathrm{Tr}_{q2}(\mathbf{K}_{12}\, E_1\, \hat{R}^n_{12})\mathbf{K}_{12}, \\ \nu\, \mathbf{K}_{12}\, E_1\, \hat{R}^n_{12} = \mathbf{K}_{12}\, \mathrm{Tr}_{q2}(\hat{R}^n_{12}\, E_1\, \mathbf{K}_{12}), \ \forall n, \end{gathered} \tag{3.10.38}$$

$$\nu\, \mathbf{K}_{12}\, E_1\, \mathbf{K}_{12} = \mathrm{Tr}_q(E)\mathbf{K}_{12}. \tag{3.10.39}$$

Calculating $\mathrm{Tr}_{q2}$ of (3.10.38), we deduce

$$\mathrm{Tr}_{q2}(\hat{R}^n_{12}\, E_1\, \mathbf{K}_{12}) = \mathrm{Tr}_{q2}(\mathbf{K}_{12}\, E_1\, \hat{R}^n_{12}), \ \ \forall n. \tag{3.10.40}$$

Further, from the first identity of (3.10.35), averaging it by means of $\mathrm{Tr}_{q2}$, we readily obtain for the algebras $\mathrm{Fun}(SO_q(N))$ and $\mathrm{Fun}(Sp_q(N))$ the analogs of (3.2.16). These take the form

$$\mathrm{Tr}_{q2}(\hat{R}^{\pm 1}_{12}) \equiv \epsilon\, \nu\, \mathrm{Tr}_2(CC^t \hat{R}^{\pm 1}_{12}) = \nu^{1 \mp 1} I_{(1)}. \tag{3.10.41}$$

Using this relation and Eq. (3.10.7), we can calculate

$$\mathrm{Tr}_q(I) = \mathrm{Tr}(D) = \nu(1 + \epsilon\, [N - \epsilon]_q) = \nu\, \mu. \tag{3.10.42}$$

We now separate irreducible representations for the left adjoint comodules (3.2.10). For an arbitrary $N \times N$ quantum matrix $E^i_j$ we have

$$E_1 = \nu^{-1} \mathrm{Tr}_{q2}(E_1\, \mathbf{K}_{12}) = E^{(0)}_1 + E^{(+)}_1 + E^{(-)}_1, \tag{3.10.43}$$

where $E^{(i)}_1 = \nu^{-1}\, \mathrm{Tr}_{q2}(\mathbf{P}^i_{12}\, E_1\, \mathbf{K}_{12}) = \nu^{-1}\, \mathrm{Tr}_{q2}(\mathbf{K}_{12}\, E_1\, \mathbf{P}^i_{12})$. It is obvious that the tensors $E^{(i)}$, $(i = \pm,\ 0)$ are invariant with respect to the adjoint coaction (3.2.10) and $\mathrm{Tr}_{q2}(\mathbf{P}^{(j)} E^{(i)} \mathbf{K}) =$

0 (if $i \neq j$) by virtue of (3.10.38). Thus, (3.10.43) is the required decomposition of the adjoint comodule $E$ into irreducible components. It is clear that the component $E^{(0)}$ is proportional to the unit matrix, $(E^{(0)})^i_j = e^{(1)} \cdot \delta^i_j$ ($e^{(1)}$ is a constant), and, thus, applying $\mathrm{Tr}_{q1}$ to (3.10.43), we obtain

$$\mathrm{Tr}_q(E) = e^{(1)}\mathrm{Tr}_q(I) = \nu\mu E^{(1)}, \tag{3.10.44}$$

where we have used the property (3.4.18), which also holds for the case of the quantum groups $SO_q(N)$ and $Sp_q(2n)$. To conclude this subsection, we note that, as in the case of the linear quantum groups, we can define fermionic and bosonic quantum hyperplanes covariant with respect to the coactions of the groups $SO_q(N)$ and $Sp_q(2n)$. Taking into account the ranks of the projectors (3.10.11) and (3.10.12), we can formulate definitions of the hyperplanes for $SO_q(N)$ ($\epsilon = 1$) and for $Sp_q(N)$ ($\epsilon = -1$) in the form

$$(\mathbf{P}^- + (\epsilon - 1)\mathbf{K})xx' = 0 \tag{3.10.45}$$

for the bosonic hyperplane (number of relations $N(N-1)/2$) and

$$(\mathbf{P}^+ + (\epsilon + 1)\mathbf{K})xx' = 0 \tag{3.10.46}$$

for the fermionic hyperplane (number of relations $N(N+1)/2$). For all these algebras, the elements $\mathbf{K}xx'$ are central elements, and it is obvious that for $Sp_q(N)$ bosons and $SO_q(N)$ fermions we have $\mathbf{K}xx' = 0$. It is interesting that the projectors $\mathbf{P}^\pm$ (3.10.5) can be represented as

$$\mathbf{P}^\pm = \frac{1}{q+q^{-1}}\left(\pm\hat{R}' + q^{\mp 1}\mathbf{1}\right) - \frac{1}{2\mu}(1 \pm \epsilon)\mathbf{K} \tag{3.10.47}$$

where the matrix

$$\hat{R}' = \hat{R} - \frac{1}{2}[\mu_-(1+\epsilon) + \mu_+(\epsilon - 1)]\mathbf{K}$$

satisfies the Hecke condition (3.1.68). However, using (3.10.28)–(3.10.35), one can directly check that $\hat{R}'$ does not obey the Yang–Baxter equation (3.1.6).

Note that the conditions (3.10.15) and (3.10.16) can be understood as conditions of invariance of the quadratic forms $x_{(1)}C^{-1}x_{(2)}$ and $y_{(1)}Cy_{(2)}$ with respect to left and right transformations of the hyperplanes $x_{(k)}$, $y_{(k)}$:

$$x^i_{(k)} \to T^i_j \otimes x^j_{(k)}, \quad y_{(k)_i} \to y_{(k)_j} \otimes T^j_i.$$

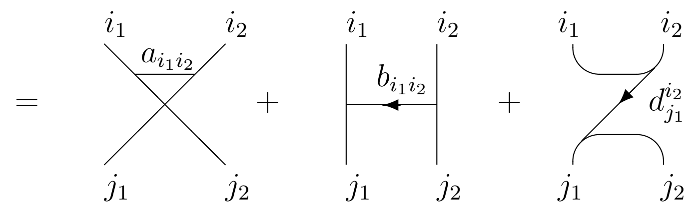
### *3.11. The multiparameter deformations $SO_{q,a_{ij}}$, $Sp_{q,a_{ij}}$ and q-supergroups $Osp_q(N|2m)$*

#### *3.11.1. General multiparametric R-matrices of the OSp type*

In this subsection, we show that it is possible to define multiparameter deformations of the quantum groups $SO_q(N)$ and $Sp_q(2n)$ and also the quantum supergroups $OSp_q(N|2m)$ (as $RTT$ algebras) if we consider for the $R$-matrix the ansatz:

$$\hat{R} = \sum_{i,j=1}^{K} a_{ij}\, e_{ij} \otimes e_{ji} + \sum_{i<j} b_{ij}\, e_{ii} \otimes e_{jj} + \sum_{i>j} d^i_j\, e_{i'j} \otimes e_{ij'} \;\Rightarrow \tag{3.11.1}$$

$$\hat{R}^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\, a_{i_1 i_2} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\, b_{i_1 i_2}\, \Theta_{i_2 i_1} + \delta^{i_1 i_2'}\delta_{j_1 j_2'}\, d^{i_2}_{j_1}\, \Theta^{i_2}_{j_1} = \tag{3.11.2}$$




where $\Theta^i{}_j := \Theta_{ij}$, $j' = K + 1 - j$; $K = N$ for the groups $SO(N)$, $K = 2n$ for the groups $Sp(2n)$, and $K = N + 2m$ for the groups $Osp(N|2m)$. The expression (3.11.2) is a natural generalization of the expression (3.6.5) for the multiparameter $R$-matrix corresponding to the linear quantum groups. Namely, the third term in (3.11.2) is constructed from the $SO$-invariant tensor $\delta^{i_1 i_2'}\delta_{j_1 j_2'}$, which takes into account the presence of the invariant metrics for the considered groups. The functions $\Theta$ are introduced in (3.11.2) (they are indicated as arrows in the graphical representation) in order to ensure that the matrix $R_{12} = P_{12}\hat{R}$ has lower triangular block form. This is necessary for the correct definition of the operators $L^{(\pm)}$ by means of the expressions (3.2.19). We demonstrate below that the ansatz (3.11.2) for the solution of the Yang–Baxter equation (3.1.6) automatically defines the family of the Birman–Murakami–Wenzl $R$-matrices with fixed parameter $\nu$ which corresponds to the quantum groups $SO_q(N)$, $Sp_q(2n)$, and $Osp_q(N|2m)$.

We substitute the ansatz (3.11.2) for the $R$-matrix in the Yang–Baxter equation (3.1.6). It is obvious that the first two terms in (3.11.2) make contributions to the Yang–Baxter equation that are analogous to the contributions of the general $R$-matrix ansatz in the case of the linear quantum groups (see Subsection **3.6**). It is, therefore, clear that for the parameters $a_{ij}$ and $b_{ij}$ we reproduce almost the same conditions (3.6.6), which in the convenient normalization $c = 1$, $b = q - q^{-1}$ have the form

$$b_{ij} = b = \lambda \;\; (\forall i, j), \;\; a_{ii} = a^0_i = \pm q^{\pm 1} \;\; (i \neq i'), \;\; a_{ij}\, a_{ji} = 1 \;\; (i \neq j, \; i \neq j'). \tag{3.11.3}$$

Note that the conditions in (3.11.3) are somewhat weaker than in (3.6.6) (because of the restrictions $i \neq i'$, $i \neq j'$). This is due to the fact that the contributions to the Yang–Baxter equation proportional to $a_{ii'}$ begin to be canceled by the contributions from the third term in (3.11.2). The corresponding condition on $a_{ii'}$ fulfilling the Yang–Baxter equation can be expressed as follows:

$$\begin{gathered} a_{jj'} = \kappa_j^{-1}(a^0_j - b), \quad a_{j'j} = \kappa_j(a^0_j - b) \quad (j \neq j') \quad \Leftrightarrow \\ a^0_j\, a_{jj'} = \kappa_j^{-1}, \quad a^0_j\, a_{j'j} = \kappa_j \quad (j \neq j'), \end{gathered} \tag{3.11.4}$$

where in addition for the constants $a^0_j$ and $\kappa_i$ we have

$$\kappa_j \kappa_{j'} = 1, \quad a^0_j = a^0_{j'}. \tag{3.11.5}$$

Taking into account Eqs. (3.11.3), the relations (3.11.4) are equivalent to the pair of possibilities $(j \neq j')$:

$$\begin{aligned} &1)\;\; a^0_j = q \quad \to \quad a_{j'j}\, \kappa_j^{-1} = q^{-1} = a_{jj'}\, \kappa_j, \\ &2)\;\; a^0_j = -q^{-1} \quad \to \quad a_{j'j}\, \kappa_j^{-1} = -q = a_{jj'}\, \kappa_j. \end{aligned} \tag{3.11.6}$$

We shall see below that if we restrict our consideration to the first possibility for all $j$ (or only the second possibility), then we obtain the $R$-matrices for the quantum groups $SO_q(N)$ and

$Sp_q(2n)$. If, however, we consider the mixed case, when both possibilities are satisfied (for different $j$), then we expect (by analogy with the linear quantum groups; see Subsection **3.6**) that the corresponding $R$-matrix will be associated with the supergroups $Osp_q(N|2m)$. The case $j = j'$ is obviously realized only for groups of the series $B$ ($SO_q(2n+1)$) and for the supergroups $Osp_q(2n+1|2m)$, and it follows from the Yang–Baxter equation (3.1.11) that

$$a_{jj'} = 1, \quad \kappa_j = 1, \quad \text{for} \;\; j = j' = \frac{K+1}{2}. \tag{3.11.7}$$

For the groups $SO_q(2n)$, $Sp_q(2n)$, and $Osp_q(2n|2m)$ the parameter $a_{jj'}$ ($j = j'$) is simply absent. Further consideration of the contributions to the Yang–Baxter equation from the third term in (3.11.2) leads to the equations

$$a_{ij}\, a_{i'j} = \kappa_j, \quad a_{ji}\, a_{ji'} = \kappa_j^{-1} \quad (\forall i \neq i'), \tag{3.11.8}$$

$$\lambda\, d^j_k\, \kappa_i + d^j_i\, d^i_k = 0 \tag{3.11.9}$$

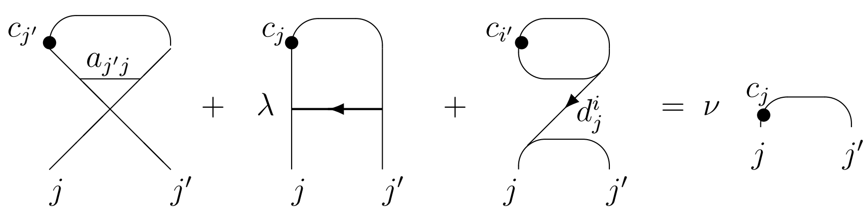
(there is no summation over repeated indices). The general solution of Eq. (3.11.9) has the form

$$d^i_j = -\lambda\, \kappa_i\, \frac{c_j}{c_i}, \tag{3.11.10}$$

where $c_i$ are arbitrary parameters. The remaining terms in the Yang–Baxter equation that do not cancel under the conditions (3.11.3)–(3.11.10) give recursion relations for the coefficients $c_i$:

$$c_{j'} a_{j'j} + \lambda c_j \Theta_{j'j} - \lambda c_j \sum_{i>j} \kappa_i \frac{c_{i'}}{c_i} = \nu c_j. \tag{3.11.11}$$

These relations can be represented graphically in the form




Another equivalent forms of (3.11.11) are

$$\sum_{k>m} d^i_{k'} d^k_j = d^i_j \left( \nu - \frac{c_{m'}}{c_m} a_{m'm} - \lambda \Theta_{m'm} \right), \tag{3.11.12}$$

$$\sum_{k<m} d^i_{k'} d^k_j = d^i_j \left( -\nu^{-1} + \frac{c_{m'}}{c_m}\, a^{-1}_{m\,m'} - \lambda \Theta_{mm'} \right), \tag{3.11.13}$$

which are related to each other by the identity

$$\sum_k d^i_{k'} d^k_j = -\lambda\, \mu\, d^i_j, \quad (\mu := (\lambda - \nu + \nu^{-1})/\lambda) \tag{3.11.14}$$

used below. Now the $R$-matrix (3.11.1) is represented in the form

$$\hat{R} = a_{ij}\, e_{ij} \otimes e_{ji} + \lambda\, \Theta_{ji}\, e_{ii} \otimes e_{jj} + \Theta_{ij}\, d^i_j\, e_{i'j} \otimes e_{ij'}, \tag{3.11.15}$$

where the summation over the indices $i, j$ is assumed and the parameters $a_{ij}$ and $d^i_j$ are fixed by the conditions (3.11.3)–(3.11.8), (3.11.10), and (3.11.11). This $R$-matrix satisfies the Yang–Baxter equation (3.1.6) and additional relations (cf. (3.10.6), (3.10.7), and (3.10.35))

$$\hat{R}^2 - \lambda \hat{R} - \mathbf{1} = -\lambda\nu\, \mathbf{K}, \quad \mathbf{K}\hat{R} = \hat{R}\mathbf{K} = \nu \mathbf{K}, \tag{3.11.16}$$

$$\mathbf{K}_{12}\, \hat{R}^{\pm 1}_{23}\, \mathbf{K}_{12} = \nu^{\mp 1}\, \mathbf{K}_{12}, \quad \mathbf{K}^2 = \mu\, \mathbf{K}, \tag{3.11.17}$$

where we have introduced the rank-1 matrix:

$$\mathbf{K} := -\lambda^{-1} \sum_{i,j} d^i_j\, e_{i'j} \otimes e_{ij'} = \sum_{i,j} \kappa_i\, \frac{c_j}{c_i}\, e_{i'j} \otimes e_{ij'} \;\; \Leftrightarrow \tag{3.11.18}$$

$$\mathbf{K}^{i_1 i_2}_{j_1 j_2} = C^{i_1 i_2}\, C_{j_1 j_2}, \quad C^{ij} = \epsilon \delta^{ij'} \frac{\kappa_j}{c_j}, \;\; C_{ij} = \frac{1}{\epsilon} \delta_{ij'} c_i. \tag{3.11.19}$$

To prove the relations (3.11.16), (3.11.17), we have used the definitions of $a_{ij}$ (3.11.3)–(3.11.7), $d^i_j$ (3.11.10) and take into account the identities (3.11.12)–(3.11.14) and

$$\Theta_{ki'}\Theta_{kj} = \Theta_{ki'}\Theta_{kj}(\Theta_{i'j} + \Theta_{ji'} + \delta_{i'j}) = \Theta_{i'j}\Theta_{ki'} + (\Theta_{ji'} + \delta_{i'j})\Theta_{kj}.$$

Thus, the $R$-matrix (3.11.15) with constraints (3.11.3)–(3.11.8), (3.11.10), and (3.11.11) automatically leads to the $R$-matrix representation of the Birman–Murakami–Wenzl algebra (the definition of this algebra is given below in Subsection **4.4**). In (3.11.19), we define the quantum metric, or quantum symplectic, matrices $C$ (cf. (3.10.13), (3.10.24)). The parameter $\epsilon$ (see Subsection **3.11**) is introduced in (3.11.19) in order to match the definition of the matrices $C$ to the study of [42], where $\epsilon = \pm 1$.

Note that the conditions (3.11.3)–(3.11.8) can be solved as

$$a_{ij} = (a^0_i)^{(\delta_{ij} - \delta_{ij'})}\, \frac{f_{ij}}{f_{ji}}, \quad \frac{f_{ij} f_{i'j}}{f_{ji} f_{ji'}} = \kappa_j \;\; (\forall i \neq i') \;\; \Rightarrow \;\; \kappa_j = \frac{f_{j'j}}{f_{jj'}}, \tag{3.11.20}$$

and, after substitution of (3.11.20) in (3.11.15), one can observe that the $R$-matrix (3.11.15) is

$$\hat{R} = \sum_{i,j} (a^0_i)^{(\delta_{ij} - \delta_{ij'})}\, \frac{f_{ij}}{f_{ji}}\, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} \frac{f_{i'i}}{f_{ii'}}\, \frac{c_j}{c_i}\, e_{i'j} \otimes e_{ij'} \tag{3.11.21}$$

and produced by the twisting (3.6.4) from the matrix

$$\hat{R} = \sum_{i,j} (a^0_i)^{(\delta_{ij} - \delta_{ij'})}\, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} \frac{\tilde{c}_j}{\tilde{c}_i}\, e_{i'j} \otimes e_{ij'}, \tag{3.11.22}$$

where $\tilde{c}_i = f_{ii'} c_i$ and the parameters $a^0_j = a^0_{j'}$, $c_j$ are determined in (3.11.6), (3.11.7), and (3.11.11). In this case, the relations (3.2.67) lead to additional conditions on the twisting parameters $f_{ij}$:

$$f_{ij} f_{i'j} = \kappa_j v_j, \quad f_{ji} f_{ji'} = v_j, \;\; \forall i, \tag{3.11.23}$$

which are consistent with (3.11.8), (3.11.20). It is evident that for $R$-matrix (3.11.22) the analogs of matrices (3.11.18), (3.11.19) are

$$\mathbf{K}^{i_1 i_2}_{j_1 j_2} = \sum_{i,j} \frac{\tilde{c}_j}{\tilde{c}_i}\, (e_{i'j})^{i_1}_{j_1} \otimes (e_{ij'})^{i_2}_{j_2} = \widetilde{C}^{i_1 i_2}\, \widetilde{C}_{j_1 j_2} \;\; \Rightarrow$$

$$\widetilde{C}^{ij} = \epsilon \delta^{ij'} \frac{1}{\tilde{c}_j}, \;\; \widetilde{C}_{ij} = \frac{1}{\epsilon} \delta_{ij'} \tilde{c}_i \;\; \Rightarrow \;\; \widetilde{C}^{ij} \, \widetilde{C}_{jk} = \delta^i_k. \tag{3.11.24}$$

Now we show that the constant $\nu$ is fixed by the relations (3.11.11) uniquely. We consider the solution of Eqs. (3.11.11), which are written in the form

$$\gamma_j a_{j'j} \kappa_j^{-1} + \lambda \Theta_{j'j} - \lambda \sum_{i=j+1}^{K} \gamma_i = \nu, \tag{3.11.25}$$

where

$$\gamma_j = \frac{c_{j'}}{c_j} \kappa_j = \frac{\tilde{c}_{j'}}{\tilde{c}_j} = \frac{1}{\gamma_{j'}}. \tag{3.11.26}$$

Equation (3.11.25) is readily solved by the changing of variables: $\gamma_i \;\; \rightarrow \;\; X_i$,

$$X_j := q^{2j} \sum_{i=j+1}^{K} \gamma_i, \;\; (X_K = 0),$$

where the inverse transformation is $\gamma_j = q^{-2j}(q^2 X_{j-1} - X_j)$ and we fix $\nu$ in (3.11.25) by taking into account the properties (3.11.26).

*3.11.2. The case of $SO_q(N)$ and $Sp_q(N)$ groups*

First, we consider the possibility 1) in (3.11.6). The possibility 2) gives, in view of a symmetry of Eq. (3.11.25), an analogous result except for the substitution $q \rightarrow -q^{-1}$. The corresponding form of Eq. (3.11.26) for $j > j'$ is

$$q \, (X_{j-1} - X_j) = q^{2j} \, \nu,$$

and we obtain the solution:

$$X_j = \nu q^{2K-1} \, \frac{1 - q^{-2(K-j)}}{1 - q^{-2}} \;\;\; \Leftrightarrow \;\; \gamma_j = \nu q^{2K-2j+1}, \;\;\; (j > j'). \tag{3.11.27}$$

For the case $K = 2n+1$ the possibility $j = j'(= K+1-j = n+1)$ is realized and Eq. (3.11.25) (in view of (3.11.7), (3.11.26), (3.11.27)) gives

$$\gamma_{n+1} = \nu q^{K-1} = 1 \;\;\; \Rightarrow \;\;\; \nu = q^{1-K}. \tag{3.11.28}$$

For the case $K = 2n$ we take $j = \frac{K}{2}$, ($j' = \frac{K}{2} + 1 > j$) in Eq. (3.11.25) and obtain $\gamma_{\frac{K}{2}} = \nu q^{K+1} - \lambda q$. On the other hand, Eq. (3.11.27) gives $\gamma_{\frac{K}{2}+1} = \nu q^{K-1}$. Thus, in view of the condition $\gamma_{\frac{K}{2}} = \gamma^{-1}_{\frac{K}{2}+1}$ (3.11.26), we deduce the equation $1 + \lambda q^K \nu = \nu^2 q^{2K}$ with two roots:

$$\nu_1 = q^{1-K}, \;\;\; \nu_2 = -q^{-1-K}. \tag{3.11.29}$$

We summarize the results (3.11.27)–(3.11.29), for the solution of (3.11.25), in the form

$$\gamma_j \equiv \frac{\tilde{c}_{j'}}{\tilde{c}_j} = \nu q^{2(N-j)+1} \; (j > j'), \;\;\; \gamma_j = 1 \; (j = j'), \;\;\; \nu = \epsilon q^{\epsilon - N} \tag{3.11.30}$$

(parameters $\tilde{c}_i$ were introduced in (3.11.22)) and relate the cases ($\epsilon = +1$) and ($\epsilon = -1$) to the groups $SO_q(N)$ and $Sp_q(N)$, respectively.

In order to determine from the conditions (3.11.30) the parameters $\tilde{c}_j$ and, thus, to fix the matrices $\widetilde{C}$ (3.11.24), we require fulfillment of the relation $\widetilde{C}^{ij} = \epsilon \widetilde{C}_{ij}$ (cf. (3.10.17)). Substitution of (3.11.24) gives the equation $\tilde{c}_j \tilde{c}_{j'} = \epsilon$, which together with (3.11.30) enables us to choose $\tilde{c}_j$ in the form [42]

$$\tilde{c}_j = \epsilon\, q^{j-\frac{1}{2}(N+\epsilon+1)} \;\; (j > j') \Rightarrow \tilde{c}_j = \epsilon_j\, q^{-\rho_j}, \tag{3.11.31}$$

where

$$\begin{gathered} \epsilon_i = +1 \; \forall i \;\; (\text{for } \; SO_q(N)), \\ \epsilon_i = +1 \; (1 \leqslant i \leqslant n), \;\; \epsilon_i = -1 \; (n+1 \leqslant i \leqslant 2n) \;\; (\text{for } \; Sp_q(2n)), \end{gathered} \tag{3.11.32}$$

and (cf. (3.10.3))

$$(\rho_1, \dots, \rho_N) = \left\{ \begin{array}{l} (n-\frac{1}{2},\, n-\frac{3}{2}, \dots, \frac{1}{2},\, 0,\, -\frac{1}{2},\, \dots, -n+\frac{1}{2})\; B : (SO_q(2n+1)), \\ (n,\, n-1,\, \dots, 1,\, -1, \dots,\, 1-n,\, -n) \quad C : (Sp_q(2n)), \\ (n-1,\, n-2, \dots, 1,\, 0,\, 0,\, -1, \dots, 1-n)\; D : (SO_q(2n)). \end{array} \right. \tag{3.11.33}$$

We note that diagonal matrices $\rho = \text{diag}(\rho_1, \dots, \rho_N)$ are equal to $\rho = \sum_{i=1}^n \delta_i\, H_i$, where elements $H_i = (e_{ii} - e_{N-i,N-i})$ form a dual basis of the Cartan subalgebras in the Lie algebras $so(N)$, $sp(2n)$ and $\delta = (\delta_1, \dots, \delta_n)$ are Weyl vectors for root systems of $so(2n), so(2n+1)$ and $sp(2n)$ (see definitions in [178] and in Subsection **3.13** below; see also [139], Subsections 3.1.1 and 3.5.2).

Thus, the final expression for the $R$-matrix (3.11.21) corresponding to the multiparameter deformation of the groups $SO(N)$ and $Sp(2n)$ (see [136, 137]) is

$$\hat{R}_{12} = \sum_{i,j} q^{(\delta_{ij} - \delta_{ij'})} \frac{f_{ij}}{f_{ji}}\, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} \frac{f_{i'i}}{f_{jj'}}\, q^{\rho_i - \rho_j}\, \epsilon_i\, \epsilon_j\, e_{i'j} \otimes e_{ij'},$$

where the parameters are defined in (3.11.23), (3.11.32), (3.11.33). The matrix $R = P\hat{R}$ is represented in the component form as

$$\begin{gathered} R^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} \left[ (q\, \delta^{i_1 i_2}|_{i_1 \neq i_2'} + q^{-1}\, \kappa_{i_1}\, \delta^{i_1 i_2'}|_{i_1 \neq i_2} + a_{i_2 i_1}|_{i_1 \neq i_2 \neq i_1'} + \delta^{i_1 i_1'} \delta^{i_2 i_2'} \right] + \\ + \lambda\, \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}\, \Theta_{i_1 i_2} - \lambda\, \kappa_{i_1} \delta^{i_1 i_2'} \delta_{j_1 j_2'}\, \Theta^{i_1}_{j_1}\, \epsilon_{i_1}\, \epsilon_{j_1} q^{\rho_{i_1} - \rho_{j_1}}, \end{gathered} \tag{3.11.34}$$

where

$$\begin{gathered} a_{ij} = 1/a_{ji} = \tfrac{f_{ij}}{f_{ji}} \;\; \forall j \neq i \neq j', \quad a_{jj'} \kappa_j = a_{j'j} \kappa_j^{-1} = q^{-1} \;\; \forall j \neq j', \\ a_{ij}\, a_{i'j} = \kappa_j, \quad a_{ji}\, a_{ji'} = \kappa_j^{-1}, \;\; \kappa_i = (\kappa_{i'})^{-1} = \tfrac{f_{i'i}}{f_{ii'}}. \end{gathered} \tag{3.11.35}$$

Now we clarify the role of the parameters $\kappa_i$. Consider the $RTT$ algebra (3.2.1) with multi-parameter $R$-matrix (3.11.34). We show that for $\kappa_i \neq \pm 1$ the element $\tau$ introduced in (3.10.25) is not central [136]. Indeed, we take the identity $\mathbf{K}_{12}\mathbf{K}_{23}\mathbf{K}_{12} = \mathbf{K}_{12}\, I_3$ (which is readily deduced from the explicit representation (3.11.18), (3.11.19)) and multiply it by $T_1 T_2 T_3$ from the right. For the right-hand side of the identity we have

$$\mathbf{K}_{12} T_1 T_2 T_3 = \tau\, \mathbf{K}_{12}\, T_3, \tag{3.11.36}$$

while for the left-hand side we obtain

$$\mathbf{K}_{12}\mathbf{K}_{23}\mathbf{K}_{12} T_1 T_2 T_3 = \mathbf{K}_{12} T_1 \mathbf{K}_{23} T_2 T_3\, \mathbf{K}_{12} = \mathbf{K}_{12} T_1 \mathbf{K}_{23}\, \mathbf{K}_{12} \tau = X_3 T_3 X_3^{-1} \mathbf{K}_{12} \tau, \tag{3.11.37}$$

where $X^i_j = C_{jk}C^{ki} = \delta^i_j\,\kappa_i$, $(X^{-1})^i_j = C_{kj}C^{ik} = \delta^i_j\,\kappa_{i'}$ and we have used the identity $T_1\mathbf{K}_{23}\,\mathbf{K}_{12} = \mathbf{K}_{23}\,\mathbf{K}_{12}X_3T_3X_3^{-1}$ followed from the definition (3.11.19). Comparing (3.11.36) and (3.11.37), we obtain

$$\tau\,T = X\,T\,X^{-1}\,\tau \quad \Rightarrow \quad \tau\,T^i_j = \kappa_i\,T^i_j\,\kappa_{j'}\,\tau.$$

Thus, only for $\kappa_i = \pm 1$ the element $\tau$ is central and one can relate the multiparameter $R$-matrices (3.11.34) with quantum deformations $SO_{q,a_{ij}}(N)$ and $Sp_{q,a_{ij}}(2n)$ of the groups $SO(N)$ and $Sp(2n)$ (see discussions after Eq. (3.10.25)).

The conditions (3.11.35) show that, for $\kappa_i = \pm 1$, the independent parameters are $q$ and $a_{ij}$ for $i < j \leqslant j'$. The numbers of these parameters are $n(n-1)/2+1$ and $n(n+1)/2+1$, respectively, for the groups of the series $C, D$ (with $N = 2n$) and $B$ (with $N = 2n+1$). Note that the last term in the square brackets in the expression (3.11.34) is appeared only for the groups of the series $B$. If we set $a_{ij} = 1$ ($j' \neq i \neq j$), $\kappa_i = 1$, then the $R$-matrices (3.11.34) are identical to the one-parameter matrices $R = P\hat{R}$ (3.10.1) deduced from (3.11.22) and given in [42].

### *3.11.3. The case of $Osp_q(N|2m)$ supergroups*

For the groups $Osp(N|2m)$ ($K = N+2m$) we choose a grading in accordance with the rules

$$\begin{array}{l} [j] = 0 \;\; \text{for} \;\; m+1 \leqslant j \leqslant m+N \\ [j] = 1 \;\; \text{for} \;\; 1 \leqslant j \leqslant m, \quad m+N+1 \leqslant j \leqslant N+2m. \end{array} \tag{3.11.38}$$

Thus, for $[j] = 0$ ($j \neq j'$) and $[j] = 1$ the possibilities 1) and 2) in (3.11.6) are respectively realized

$$a^0_j = (-1)^{[j]}\,q^{1-2[j]} = (-1)^{[j]}\,q^{(-1)^{[j]}}, \quad [j] = (j'). \tag{3.11.39}$$

In this case, Eq. (3.11.25) is written as the system of equations

$$\gamma_j\,q + \lambda\sum_{i=j+1}^{K}\gamma_i = -\nu, \;\; (N+m+1 \leqslant j \leqslant N+2m), \tag{3.11.40}$$

$$\gamma_j\,q - \lambda + \lambda\sum_{i=j+1}^{K}\gamma_i = -\nu, \;\; (1 \leqslant j \leqslant m), \tag{3.11.41}$$

$$\gamma_j\,q^{(\delta_{j'j}-1)} + \lambda\Theta_{j'j} - \lambda\sum_{i=j+1}^{K}\gamma_i = \nu, \;\; (m+1 \leqslant j \leqslant m+N). \tag{3.11.42}$$

In (3.11.42), for the case $j = j'$, we take into account (3.11.7). The solution of (3.11.40) is (cf. (3.11.27)):

$$\gamma_j = -\nu\,q^{2(j-K)-1}, \;\; (N+m+1 \leqslant j \leqslant N+2m), \tag{3.11.43}$$

and we have $\lambda\sum\limits_{i>m+N}\gamma_i = \nu(q^{-2m}-1)$. Using this fact, Eq. (3.11.42) is written in the form

$$\gamma_j q^{(\delta_{j'j}-1)} + \lambda\Theta_{j'j} - \lambda\sum_{i=j+1}^{m+N}\gamma_i = \nu q^{-2m}, \tag{3.11.44}$$

and its solution is

$$\gamma_j = \nu\,q^{2(N-j)+1}, \;\; (j' < j \leqslant N+m). \tag{3.11.45}$$

In addition, for $N = 2n+1$ and $j = j' = m+n+1$ we have

$$\gamma_{m+n+1} = \nu q^{N-2m-1} = 1, \quad \nu = q^{2m-N+1}, \tag{3.11.46}$$

and for $N = 2n$ we obtain the condition $\gamma_{m+\frac{N}{2}} = \nu q^{N-2m+1} - \lambda q = \gamma^{-1}_{m+\frac{N}{2}+1}$, which is equivalent to the quadratic equation on $\nu$:

$$(\nu q^{N-2m} - q)(\nu q^{N-2m} + q^{-1}) = 0. \tag{3.11.47}$$

Accordingly, we summarize the results (3.11.43), (3.11.45)–(3.11.47) as

$$\gamma_j \equiv \frac{\tilde{c}_{j'}}{\tilde{c}_j} = (-1)^{[j]} \nu q^{(-1)^{[j]}(2N-2j+1)-[j]4m} \;\; (j > j'), \;\; \nu = \epsilon q^{\epsilon+2m-N}, \tag{3.11.48}$$

where $\epsilon = \pm 1$ and the case $\epsilon = +1$ corresponds to $Osp_q(N|2m)$, while the case $\epsilon = -1$ we relate to a quantum group denoted as $Osp'_q(2m|2n)$. It is obvious that for the groups $Osp_q(2n+1|2m)$ (as well as for $SO_q(2n+1)$) we have $\gamma_j = \gamma_{j'} = 1$ for $j = j'$. Note that if in (3.11.48) we set $m = 0$, or $N = 0$ and $q \to -q^{-1}$, then we reproduce (3.11.30).

The analog of the relation (3.10.17) for the groups $Osp_q(N|2m)$ is the equation

$$\widetilde{C}^{ij} = (-1)^{(i)} \, \epsilon \, \widetilde{C}_{ij},$$

which is equivalent to $(-1)^{(i)} \, \tilde{c}_i \, \tilde{c}_{i'} = \epsilon$, and taking into account (3.11.48), we obtain

$$c_j = \epsilon \, q^{(-1)^{[j]}(j-m-N-\frac{1}{2})-\frac{\epsilon}{2}+\frac{N}{2}} \;\; (j > j') \;\; \Rightarrow \;\; \tilde{c}_j = \epsilon_j q^{-\rho_j},$$

where $[j] = 0, 1$ is the grading (3.11.38). The parameters $(\rho_1, \ldots, \rho_K)$, $(\epsilon_1, \ldots, \epsilon_K)$ are fixed according to the following cases:

1) The case $\epsilon = +1$, $\nu = q^{1+2m-N}$ for $Osp_q(N|2m)$ $(N = 2n+1)$:

$$\begin{gathered} \rho_i = (\underbrace{\frac{N}{2} - m, \ldots, \frac{N}{2} - 1}_{m}; \underbrace{\frac{N}{2} - 1, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, 1 - \frac{N}{2}}_{2n+1}; \underbrace{1 - \frac{N}{2}, \ldots, m - \frac{N}{2}}_{m}) \\ \epsilon_i = (\underbrace{-1, \ldots, -1}_{m}; \underbrace{+1, \ldots, +1}_{2n+1}; \underbrace{+1, \ldots, +1}_{m}) \end{gathered} \tag{3.11.49}$$

2) The case $\epsilon = +1$, $\nu = q^{1+2m-N}$ for $Osp_q(N|2m)$ $(N = 2n)$:

$$\begin{gathered} \rho_i = (\underbrace{n - m, \ldots, n - 1}_{m}; \underbrace{n - 1, \ldots, 1, 0, 0, -1, \ldots, 1 - n}_{2n}; \underbrace{1 - n, \ldots, m - n}_{m}) \\ \epsilon_i = (\underbrace{-1, \ldots, -1}_{m}; \underbrace{+1, \ldots, +1}_{2n}; \underbrace{+1, \ldots, +1}_{m}) \end{gathered} \tag{3.11.50}$$

3) The case $\epsilon = -1$, $\nu = -q^{-1+2m-2n}$ for $Osp'_q(2m|2n)$:

$$\begin{gathered} \rho_i = (\underbrace{n + 1 - m, \ldots, n}_{m}; \underbrace{n, \ldots, 1, -1, \ldots, -n}_{2n}; \underbrace{-n, \ldots, m - 1 - n}_{m}) \\ \epsilon_i = (\underbrace{-1, \ldots, -1}_{m}; \underbrace{+1, \ldots, +1}_{n}, \underbrace{-1, \ldots, -1}_{n}; \underbrace{-1, \ldots, -1}_{m}) \end{gathered} \tag{3.11.51}$$

To conclude this subsection, we give the final expression for the $R$-matrix (3.11.21) corresponding to the quantum supergroups $Osp_q(N|2m)$ $(\epsilon = +1)$ and $Osp'_q(2m|2n)$ $(\epsilon = -1)$:

$$\hat{R}_{12} = \sum_{i,j} (-1)^{(i)[j]} \, q^{(-)^{(i)}(\delta_{ij} - \delta_{ij'})} \, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} q^{\rho_i - \rho_j} \, \epsilon_i \, \epsilon_j \, e_{i'j} \otimes e_{ij'}, \tag{3.11.52}$$

where the parameters $\epsilon_i$, $\rho_j$ are defined in (3.11.49)–(3.11.51). We stress here that the matrix units $e_{ij}$ and tensor products in (3.11.52) *are not graded*, as follows form the discussion of the general Yang–Baxter solution (3.11.21) in Subsection **3.11.1**. To obtain (3.11.52), we used the condition (3.11.39) and put $f_{ij}/f_{ji} = (-1)^{[i][j]}$ $(\forall i \neq j \neq i')$, $f_{i'i} = f_{ii'} = 1$ in (3.11.21). This choice of the parameters $f_{ij}$ is such that $\hat{R}_{12}$ tends to the supertransposition matrix $(-1)^{(1)(2)}P_{12}$ when $q \to 1$ (for the notation $(-1)^{(1)(2)}$ see (3.7.6)). In the supergroup case, the multiparameter $R$-matrices are restored directly from (3.11.52) by the same twisting (3.6.4) if we take into account conditions (3.11.23).

The quantum supergroup $Osp_q(N, 2m)$ is the graded algebra generated by elements $T^i_{\ k}$ $(i, k = 1, \ldots, N + 2m)$ of $(N + 2m) \times (N + 2m)$ supermatrix. As in the case of quantum supergroups $GL_q(N, M)$ and $SL_q(N, M)$, the generators $\{T^i_{\ k}\}$ of the $Osp_q(N, 2m)$ algebra satisfy the graded $RTT$ relations (3.7.14), (3.7.15) but with the $Osp_q(N, 2m)$-type $R$-matrix (3.11.52). The Hopf structure of the quantum supergroup $Osp_q(N, 2m)$ is introduced in the same way as the Hopf structure of $GL_q(N, M)$ (see Subsection **3.7**).

We note that the parameters $\nu$ for the cases $Osp_q$ (3.11.50) and $Osp'_q$ (3.11.51) are related to each other by means of the transformation: $q \leftrightarrow -q^{-1}$, $n \leftrightarrow m$. However, this transformation does not relate the corresponding $R$-matrices (3.11.52). Our conjecture is that for the cases $Osp_q$ (3.11.50) and $Osp'_q$ (3.11.51) the $R$-matrices (3.11.52) and corresponding quantum groups are inequivalent.

The $R$-matrices constructed in this subsection for the quantum supergroups realize $R$-matrix representations of the Birman–Murakami–Wenzl algebra, since they are the special examples of the general $R$-matrix (3.11.15) which satisfy (3.11.16), (3.11.17). Some of these $R$-matrices can be obtained on the basis of the results of [161], in which Baxterized trigonometric solutions (see next Subsection **3.12**) of the Yang–Baxter equation associated with the classical supergroups $Osp(N|2m)$ were obtained. Rational solutions, some special cases, and other questions relating to the subject of the quantum supergroups $Osp_q(N|2m)$ are also discussed in [162, 164] and [167, 168].

### *3.12. $SO_q(N)$-, $Sp_q(2n)$- and $Osp_q(N|2m)$-invariant Baxterized R-matrices*

Arguing, as in Subsection **3.8**, we conclude that the $SO_q(N)$- and $Sp_q(N)$- (as well as $Osp_q(N|2m)$-) invariant Baxterized matrices $\hat{R}(x)$ must be sought (by virtue of the fact that the characteristic equation (3.10.4) is cubic) in the form of a linear combination of the three basis matrices $\mathbf{1}$, $\hat{R}$, $\hat{R}^2$. Expressing $\hat{R}^2$ in terms of $\mathbf{K}$ and $\hat{R}$, we can represent invariant $R(x)$-matrix in the form [84]

$$\hat{R}(x) = c(x)\left(\mathbf{1} + a(x)\hat{R} + b(x)\mathbf{K}\right), \tag{3.12.1}$$

where $a(x)$, $b(x)$, and $c(x)$ are certain functions that depend on the spectral parameter $x$. We determine the functions $a(x)$, $b(x)$ from the Yang–Baxter equation (3.8.2). The normalizing function $c(x)$ is not fixed by Eq. (3.8.2). After substitution of (3.12.1) in (3.8.2) and using (3.10.28)–(3.10.35), the following relations arise [84]:

$$\begin{aligned}
&a_1 + a_3 + \lambda a_1\, a_3 = a_2, \\
&b_3 - b_2 - \lambda\nu a_1\, a_3 + \nu a_1\, b_3 - \lambda a_1\, b_2\, b_3 + \lambda^2 a_1\, a_3\, b_2 + \\
&+ b_1(1 + \nu a_3 - \lambda a_3\, b_2 + \mu b_3 + \nu^{-1} a_2\, b_3 + b_2\, b_3) = 0, \\
&a_2\, b_1 + a_3\, b_1\, b_2 = a_1\, b_2 + \lambda a_1\, a_3\, b_2, \quad a_2\, b_3 + a_1\, b_2\, b_3 = a_3\, b_2 + \lambda a_1\, a_3\, b_2,
\end{aligned} \tag{3.12.2}$$

where we denoted

$$a_1 = a(x), \quad a_2 = a(xy), \quad a_3 = a(y), \quad b_1 = b(x), \quad b_2 = b(xy), \quad b_3 = b(y).$$

The four relations (3.12.2) are equivalent to the three functional equations

$$a(x) + a(y) + \lambda a(x)a(y) = a(xy), \tag{3.12.3}$$

$$\begin{gathered} b(y) - b(xy) + a(x)[\nu b(y) - \lambda\nu a(y) - \lambda b(xy)b(y) + \lambda^2 a(y)b(xy)] + \\ + b(x)[1 + \nu a(y) - \lambda a(y)b(xy) + \mu b(y) + \nu^{-1}a(xy)b(y) + b(xy)b(y)] = 0, \end{gathered} \tag{3.12.4}$$

$$a(xy)b(y) + a(x)b(xy)b(y) = b(xy)(a(y) + \lambda a(x)a(y)), \tag{3.12.5}$$

since the third and fourth relations in (3.12.2) give the same equation (3.12.5). As was to be expected, Eq. (3.12.3) is identical to Eq. (3.8.3) obtained in the $GL_q(N)$ case, and its general solution is given in (3.8.4). By means of (3.12.3), we can transform the right-hand side of Eq. (3.12.5) in such a way that (3.12.5) reduces to the equation

$$\frac{a(x)}{a(xy)} = \frac{b(xy) - b(y)}{b(xy)(b(y)+1)} \equiv 1 - \frac{b(y)(1+b(y))^{-1}}{b(xy)(1+b(xy))^{-1}}. \tag{3.12.6}$$

We now note that Eq. (3.12.3) can be rewritten in the form

$$\frac{a(x)}{a(xy)} = 1 - \frac{a(y)(\lambda a(y)+1)^{-1}}{a(xy)(\lambda a(xy)+1)^{-1}} \tag{3.12.7}$$

and, comparing (3.12.6) and (3.12.7), we arrive at the result

$$\frac{a(y)(b(y)+1)}{(\lambda a(y)+1)b(y)} = \text{const} \equiv \frac{\alpha+1}{\lambda}, \tag{3.12.8}$$

where $\alpha$ denotes an arbitrary parameter. The specific choice of the form of the constant in the right-hand side of (3.12.8) is made for convenience in what follows. Substituting the solution (3.8.4) in (3.12.8), we obtain the following general expression for $b(y)$:

$$b(y) = \frac{y^\xi - 1}{\alpha y^\xi + 1}. \tag{3.12.9}$$

It is a remarkable fact that Eq. (3.12.4) is satisfied identically on the functions (3.8.4) and (3.12.9) if the constant $\alpha$ satisfies the quadratic equation

$$\alpha^2 - \frac{\lambda}{\nu}\alpha - \frac{1}{\nu^2} = 0. \tag{3.12.10}$$

The two solutions of this equation are readily found:

$$\alpha_\pm = \pm\frac{q^{\pm 1}}{\nu}, \tag{3.12.11}$$

where we recall that

$$\begin{aligned} &\nu = \epsilon q^{\epsilon - N} && \text{for groups } SO_q(N) \ (\epsilon=+1), \ Sp_q(N) \ (\epsilon=-1); \\ &\nu = \epsilon q^{\epsilon+2m-N} && \text{for supergroups } Osp_q(N|2m) \ (\epsilon=+1) \text{ and} \\ &&& \qquad\qquad Osp'_q(2m|N) \ (N=2n, \ \epsilon=-1). \end{aligned} \tag{3.12.12}$$

Thus, the solutions of the Yang–Baxter equation (3.8.2) can be represented in the form[19] [84, 237]

$$\hat{R}(x) = c(x)\left(\mathbf{1} + \frac{1}{\lambda}(x^\xi - 1)\hat{R} + \frac{x^\xi - 1}{\alpha x^\xi + 1}\mathbf{K}\right) \tag{3.12.13}$$

and we have the two possibilities $\alpha = \alpha_\pm$ (3.12.11), which are inequivalent (both for all the cases $SO_q(N)$, $Sp_g(N)$, and $Osp_q(N|2m)$), since these solutions cannot be reduced to each other by any functional transformations of the spectral parameter $x$. However, these solutions are related by the transformation $q \to -q^{-1}$. For convenience, we choose $c(x) = x$ and $\xi = -2$ in (3.12.13); then for the $R$-matrices (3.12.13) we can propose four equivalent expressions:

$$\hat{R}^\pm(x) := \frac{1}{\lambda}\left(x^{-1}\hat{R} - x\hat{R}^{-1}\right) + \frac{\alpha_\pm + 1}{\alpha_\pm x^{-1} + x}\mathbf{K} = \tag{3.12.14}$$

$$= \frac{(q^{\pm 2}x^{-1} - x)}{(q^{\pm 2} - 1)\,x^2}\,\frac{(\hat{R} \pm q^{\pm 1}x^2)}{(\hat{R} \pm q^{\pm 1}x^{-2})} = \tag{3.12.15}$$

$$= \frac{x - x^{-1}}{\lambda(x + \alpha_\pm x^{-1})}\left(-x\hat{R}^{-1} - \alpha_\pm x^{-1}\hat{R} + \frac{\lambda(\alpha_\pm + 1)}{x - x^{-1}}\right) =$$

$$= \frac{(x^{-1}q - xq^{-1})}{\lambda}\mathbf{P}^+ + \frac{(xq - (xq)^{-1})}{\lambda}\mathbf{P}^- + \frac{(q^{\pm 2}x^{-1} - x)}{(q^{\pm 2} - 1)}\,\frac{(x^{-1} + x\,\alpha_\pm)}{(x + x^{-1}\,\alpha_\pm)}\mathbf{P}^0, \tag{3.12.16}$$

where projectors $\mathbf{P}^\pm$ and $\mathbf{P}^0$ are defined in (3.10.5). The last expression is the spectral decomposition of $\hat{R}(x)$, from which, for example, we can readily obtain the identities

$$\hat{R}^+(\pm q) = \pm(q + q^{-1})\,\mathbf{P}^-, \qquad \hat{R}^-(\pm q^{-1}) = \pm(q + q^{-1})\,\mathbf{P}^+, \tag{3.12.17}$$

$$\lim_{x^2 \to -\alpha_\pm} R^\pm(x) \sim \mathbf{P}^0, \qquad \hat{R}^\pm(1) = \mathbf{1}, \qquad \hat{R}^\pm(i) = \pm\frac{i(q + q^{-1})}{\lambda}(\mathbf{1} - 2\mathbf{P}^\pm). \tag{3.12.18}$$

From rational representations (3.12.15) of $R$-matrix, one can immediately deduce the identity

$$\hat{R}^\pm(x)\,\hat{R}^\pm(x^{-1}) = \left(1 - \frac{(x - x^{-1})^2}{\lambda^2}\right)\cdot\mathbf{1}. \tag{3.12.19}$$

Note that the relations (3.12.17)–(3.12.19) agree with the Yang–Baxter equation (3.8.2).

The cross-unitarity condition for the BMW-type $R$-matrix (3.12.14) can be written in the matrix form as (cf. (3.8.9))

$$\begin{aligned}
&\mathrm{Tr}_{D(2)}\left(\hat{R}^\pm_1(x)P_{01}\hat{R}^\pm_1(z)\right) = \eta^\pm(x)\,\eta^\pm(z)\,D_0\,I_1, \\
&\mathrm{Tr}_{Q(1)}\left(\hat{R}^\pm_1(x)P_{23}\hat{R}^\pm_1(z)\right) = \eta^\pm(x)\,\eta^\pm(z)\,Q_3\,I_2,
\end{aligned} \tag{3.12.20}$$

where the matrices $D, Q$ are defined in (3.1.20) and

$$(x\,z)^2 = \alpha_\pm^2, \qquad \eta^\pm(x) = \frac{1}{\lambda}(x - x^{-1})\frac{(\alpha_\pm \nu x^2 + \nu^{-1})}{(x^2 + \alpha_\pm)}, \qquad \alpha_\pm := \pm\frac{q^{\pm 1}}{\nu}.$$

[19] The Baxterized trigonometric $R$-matrices (3.12.13), corresponding to the one of the parameter choice in (3.12.11), were first found by V. Bazhanov in 1984 and were published in [169, 170]. The same $R$-matrices were independently constructed in [171].

The Baxterized $\hat{R}$-matrices (3.12.13) and (3.12.14)–(3.12.16) determine algebras with the defining relations (3.9.1). However, a realization of the operators $L(x)$ in terms of the generators $L^{(\pm)}$ of the quantum algebras $U_q(so(N))$ and $U_q(sp(N))$ (analogous to (3.9.2)) is unfortunately missing (see, however, [101, 102] and discussion of the case $q \to 1$ in [103, 104]).

To conclude this subsection, we present the expressions for the rational $R$-matrices of the Yangians $Y(so(N))$, $Y(sp(N))$, and $Y(osp(N|2m))$. We give the definition of Yangians in Subsection **3.13** below. We make the ansatz $x = \exp(-\lambda\theta/2)$ for the spectral parameter in (3.12.14) and rewrite the $R$-matrix in the form (cf. (3.9.14))

$$\begin{gathered}\hat{R}(\theta) := \hat{R}\left(e^{-\frac{\lambda}{2}\theta}\right) = \cosh(\lambda\theta/2)\,[\mathbf{1}-\mathbf{K}] + \tfrac{1}{\lambda}\sinh(\lambda\theta/2)\,[\hat{R}+\hat{R}^{-1}] + \\ + [\cosh(\lambda\theta/2)+\beta_\pm\sinh(\lambda\theta/2)]^{-1}\,\mathbf{K},\end{gathered} \tag{3.12.21}$$

where $\beta_\pm = \dfrac{\alpha_\pm - 1}{\alpha_\pm + 1}$. The Yangian $R$-matrices can be obtained from (3.12.21) after the passage to the limit $h \to 0$ ($q = \exp(h) \to 1$). Further, it is easy to see that the cases $\alpha = \alpha_+$, $\epsilon = 1$ ($SO_q(N)$) and $\alpha = \alpha_-$, $\epsilon = -1$ ($Sp_q(2n)$) are reduced to the $GL(N)$-symmetric Yang's $R$-matrix (3.9.16). The nontrivial $SO(N)$- and $Sp(N)$-symmetric Yangian $R$-matrices for $Y(so(N))$ and $Y(sp(N))$ correspond to the choice

$$\alpha = \alpha_-, \quad \epsilon = 1 \quad (SO_q(N))\,; \qquad \alpha = \alpha_+, \quad \epsilon = -1 \quad (Sp_q(N)) \tag{3.12.22}$$

and have the form

$$\hat{R}(\theta) = (\mathbf{1} + \theta\, P_{12}) + \frac{2\,\theta}{(2\epsilon - (N+2\theta))}\, K^{(0)}_{12}. \tag{3.12.23}$$

The matrix $K^{(0)}_{12}$ is defined in (3.10.9). Nontrivial rational $R$-matrices for super Yangians $Y(osp(N|2m))$ and $Y'(osp(2m|2n))$ can be obtained from (3.12.21) in the cases:

$$\alpha = \alpha_-, \quad \epsilon = 1 \quad (Osp_q(N|2m))\,; \qquad \alpha = \alpha_+, \quad \epsilon = -1, \;\; N = 2n \quad (Osp'_q(2m|2n)).$$

The form of these supersymmetric $R$-matrices is

$$\hat{R}(\theta) = (\mathbf{1} + \theta\,\mathcal{P}_{12}) + \frac{2\,\theta}{2\epsilon + 2m - (N + 2\,\theta)}\,\mathcal{K}^{(0)}_{12}, \tag{3.12.24}$$

where $\mathcal{P}^{i_1 i_2}_{j_1 j_2} = (-1)^{[i_1][i_2]}\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the supertransposition operator (the parity $[j]$ is defined in (3.11.38)). The matrix $(\mathcal{K}^{(0)})^{i_1 i_2}_{j_1 j_2} = \mathcal{C}^{i_1 i_2}\mathcal{C}_{j_1 j_2}$ is a classical limit ($q \to 1$) of the rank-1 matrix $\mathbf{K}$ in the supersymmetric case and the ortho-symplectic matrices $\mathcal{C}^{ij} = \epsilon_j \delta^{ij'}$, $\mathcal{C}_{ij} = \epsilon_i \delta_{ij'}$ are determined by their parameters $\epsilon_i$ (3.11.49)–(3.11.51). Then the defining relations for the generators (3.9.17) of the Yangians $Y(so(N))$, $Y(sp(N))$ and $Y(osp(N|2m))$, $Y'(osp(2m|2n))$ are identical to (3.9.13) and (3.9.21), respectively, while the comultiplication is given by (3.9.18).

The Yangian $R$-matrix (3.12.23) for the $SO(N)$ case was found in [4, 5] and that for the $Sp(2n)$ case — in [166]. These $R$-matrices were used in [103] to construct and investigate exactly solvable $SO(N)$- and $Sp(2n)$-symmetric magnets. Twisted Yangians for the $SO(N)$ and $Sp(2n)$ cases have been considered in [144, 145]. The super Yangians of the type $Y(osp(N|2m))$ and corresponding spin chain models were discussed in [167, 168].

### *3.13. Split Casimir operators and rational solutions of Yang–Baxter equations. Yangians*

The material of this subsection is based on the papers [143, 172]; see also [173, 174, 176, 177].

*3.13.1. Invariant R-matrices for simple Lie algebras $\mathfrak{g}$*

Let $\mathfrak{g}$ be a simple Lie algebra with the basis elements $X_a|_{a=1,\dots,\dim(\mathfrak{g})}$ and defining relations

$$[X_a,\ X_b] = X^d_{ab}\, X_d, \tag{3.13.1}$$

where $X^d_{ab}$ are the structure constants. We denote an enveloping algebra of the Lie algebra $\mathfrak{g}$ as $\mathcal{U}(\mathfrak{g})$. Let $\mathsf{g}^{df}$ be the inverse matrix to the Cartan–Killing metric

$$\mathsf{g}_{ab} \equiv X^d_{ac}\, X^c_{bd} = \mathrm{Tr}(\mathrm{ad}(X_a)\cdot \mathrm{ad}(X_b)), \tag{3.13.2}$$

where ad denotes adjoint representation. Introduce the operator

$$\hat{C} = \mathsf{g}^{ab} X_a \otimes X_b \equiv X_a \otimes X^a \;\; \in \;\; \mathfrak{g}\otimes\mathfrak{g} \;\; \subset \;\; \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}), \tag{3.13.3}$$

which is called the *split (or polarized) Casimir operator* of the Lie algebra $\mathfrak{g}$. The operator (3.13.3) is related to the usual quadratic Casimir operator

$$C_{(2)} = \mathsf{g}^{ab}\, X_a \cdot X_b \;\; \in \;\; \mathcal{U}(\mathfrak{g}) \tag{3.13.4}$$

by means of the formula

$$\Delta(C_{(2)}) = C_{(2)} \otimes I + I \otimes C_{(2)} + 2\,\hat{C}, \tag{3.13.5}$$

where $\Delta\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is the standard comultiplication defined by $\Delta(X_a) = X_a \otimes I + I \otimes X_a$. Let $u$ be a spectral parameter. One can check that the operator function

$$r(u) = \frac{\hat{C}}{u} = \frac{X_a \otimes X^a}{u} \equiv r_{21}(u), \tag{3.13.6}$$

obeys the semiclassical Yang–Baxter equation (cf. (3.3.2)):

$$[r_{12}(u),\, r_{13}(u+v)] + [r_{13}(u+v),\, r_{23}(v)] + [r_{12}(u),\, r_{23}(v)] = 0. \tag{3.13.7}$$

The aim of this subsection is to find rational (as a function in the spectral parameter $u$) solutions $R(u)$ of the Yang–Baxter equations (3.9.12) ($\theta' = u$, $\theta = u+v$):

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u), \tag{3.13.8}$$

that are unitary

$$R_{12}(u) R_{21}(-u) = \mathbf{1} \equiv I \otimes I \tag{3.13.9}$$

and possess semiclassical behavior $R_{12}(u) \to \mathbf{1}$ as $u \to \infty$. We then write the expansion

$$R_{12}(u) = \mathbf{1} + \frac{\hat{C}}{u} + \frac{X}{u^2} + O\Big(\frac{1}{u^3}\Big). \tag{3.13.10}$$

The second term here is justified by (3.13.7). As we will see below, the solutions of this kind are given (up to a renormalization) by (3.12.23) for Lie algebras $\mathfrak{g} = \mathfrak{so}(N)$ and $\mathfrak{sp}(N)|_{N=2r}$ in defining representations.

First, we use the unitarity condition (3.13.9) to find $X$:

$$\mathbf{1} = R_{12}(u) R_{21}(-u) = \mathbf{1} - \frac{1}{u^2}\hat{C}^2 + \frac{1}{u^2}(X_{12} + X_{21}) + \dots.$$

We search for the symmetric solutions $X_{12} = X_{21}$, so we have

$$X_{12} = \frac{1}{2}\hat{C}^2. \tag{3.13.11}$$

Then we examine the limit $v \to \infty$ of (3.13.8):

$$\begin{gathered} R_{12}(u)\left(\mathbf{1} + \frac{\hat{C}_{13}}{u+v} + \frac{X_{13}}{(u+v)^2} + \ldots\right)\left(\mathbf{1} + \frac{\hat{C}_{23}}{v} + \frac{X_{23}}{v^2} + \ldots\right) = \\ = \left(\mathbf{1} + \frac{\hat{C}_{23}}{v} + \frac{X_{23}}{v^2} + \ldots\right)\left(\mathbf{1} + \frac{\hat{C}_{13}}{u+v} + \frac{X_{13}}{(u+v)^2} + \ldots\right) R_{12}(u). \end{gathered} \tag{3.13.12}$$

We expand out the brackets in (3.13.12) and multiply both sides by $v(u+v)$. As a result, we obtain

$$\begin{gathered} R_{12}(u)\left(v(\hat{C}_{23} + \hat{C}_{13}) + u\hat{C}_{23} + \hat{C}_{13}\hat{C}_{23} + \tfrac{u+v}{v}X_{23} + \tfrac{v}{u+v}X_{13} + \ldots\right) = \\ = \left(v(\hat{C}_{23} + \hat{C}_{13}) + u\hat{C}_{23} + \hat{C}_{23}\hat{C}_{13} + \tfrac{u+v}{v}X_{23} + \tfrac{v}{u+v}X_{13} + \ldots\right) R_{12}(u). \end{gathered} \tag{3.13.13}$$

We see that the terms of order $v$ give

$$[R_{12}(u),\, (\hat{C}_{23} + \hat{C}_{13})] = 0 \quad \Rightarrow \quad [R_{12}(u),\, I \otimes X_a + X_a \otimes I] = 0, \tag{3.13.14}$$

which is the condition of the invariance of $R(u)$ under the action of $\mathfrak{g}$. Thus, by Schur's lemma, one can express the image $R_{(\mu\nu)}(u) = (T_\mu \otimes T_\nu)R(u)$ of the operator $R(u)$ in the representation $(T_\mu \otimes T_\nu)$ of $\mathfrak{g}$ as follows:

$$R_{(\mu\nu)}(u) = \sum_{T_\lambda \subset T_\mu \otimes T_\nu} \tau_\lambda(u)\; P_\lambda, \tag{3.13.15}$$

where $P_\lambda$ is the projector onto the irreducible subrepresentation $T_\lambda \subset T_\mu \otimes T_\nu$ and $\tau_\lambda(u)$ are some rational functions of $u$, which is yet undetermined. At this stage, we require that the set of projectors $P_\lambda$ form the complete system of mutually orthogonal projectors

$$\sum_\lambda P_\lambda = I_\mu \otimes I_\nu, \qquad P_\lambda\, P_{\lambda'} = P_\lambda \delta_{\lambda\lambda'}. \tag{3.13.16}$$

We also require that the decomposition $T_\mu \otimes T_\nu = \sum_\lambda T_\lambda$ be without multiplicities, otherwise, $R(u)$ acts on the isomorphic components $T_{\lambda_1}, \ldots, T_{\lambda_r}$ as matrix $||M_{ij}(u)||_{i,j=1,\ldots,r}$ which is not, in general, diagonalizable.

The terms of order $v^0 = 1$ in (3.13.13) give

$$R_{12}(u)\left(u\hat{C}_{23} + \hat{C}_{13}\hat{C}_{23} + X_{23} + X_{13}\right) = \left(u\hat{C}_{23} + \hat{C}_{23}\hat{C}_{13} + X_{23} + X_{13}\right) R_{12}(u). \tag{3.13.17}$$

We rewrite it by using (3.13.11) and applying the identities

$$\begin{gathered} \hat{C}_{13}\,\hat{C}_{23} + \tfrac{1}{2}(\hat{C}_{13}^2 + \hat{C}_{23}^2) = \tfrac{1}{2}[\hat{C}_{13},\ \hat{C}_{23}] + \tfrac{1}{2}(\hat{C}_{13} + \hat{C}_{23})^2, \\ [\hat{C}_{13},\ \hat{C}_{23}] = X^a_{bc}\, X^b \otimes X^c \otimes X_a = -\tfrac{1}{2}[\Delta C_{(2)},\ (I \otimes X_a)] \otimes X^a, \end{gathered}$$

so that, because of (3.13.14), we simplify (3.13.17) as

$$R_{12}(u)\left(u\hat{C}_{23} + \frac{1}{2}[\hat{C}_{13},\ \hat{C}_{23}]\right) = \left(u\hat{C}_{23} + \frac{1}{2}[\hat{C}_{23},\ \hat{C}_{13}]\right) R_{12}(u) \quad \Rightarrow$$

$$R_{12}(u)\left(u(I\otimes X_a)-\tfrac{1}{4}[\Delta C_{(2)},\,I\otimes X_a]\right)=\left(u(I\otimes X_a)+\tfrac{1}{4}[\Delta C_{(2)},\,I\otimes X_a]\right)R_{12}(u). \tag{3.13.18}$$

Now we consider the image of (3.13.18) in the representation $T_\mu\otimes T_\nu$, substitute (3.13.15) and act by projectors $P_\kappa$ and $P_\lambda$ from the right and the left, respectively. As a result, we deduce the relation between coefficients $\tau_\lambda(u)$:

$$\begin{aligned}&\tau_\lambda(u)\left(u-\tfrac{1}{4}\big(C_{(2)}(\lambda)-C_{(2)}(\kappa)\big)\right)P_\lambda\,(I\otimes X_a)P_\kappa=\\&=\tau_\kappa(u)\left(u+\tfrac{1}{4}\big(C_{(2)}(\lambda)-C_{(2)}(\kappa)\big)\right)P_\lambda\,(I\otimes X_a)P_\kappa,\end{aligned} \tag{3.13.19}$$

where $C_{(2)}(\lambda)$ is the value of the quadratic Casimir operator (3.13.4) in the representation $T_\lambda$ and for brevity we write $(I\otimes X_a)$ instead of $(T_\mu(I)\otimes T_\nu(X_a))$. We enumerate the representations $T_\lambda$ of the Lie algebra $\mathfrak{g}$ by their highest weights $\lambda$. In this case, the value of $C_{(2)}(\lambda)$ is given by the formula

$$C_{(2)}(\lambda)=(\lambda,\lambda+2\,\delta), \tag{3.13.20}$$

where $\delta$ is the Weil vector of the algebra $\mathfrak{g}$. Finally, from Eq. (3.13.19), in the case when $P_\lambda\,(I\otimes X_a)P_\kappa\neq 0$, we have

$$\frac{\tau_\lambda(u)}{\tau_\kappa(u)}=\frac{u+\frac{1}{4}\big(C_{(2)}(\lambda)-C_{(2)}(\kappa)\big)}{u-\frac{1}{4}\big(C_{(2)}(\lambda)-C_{(2)}(\kappa)\big)}. \tag{3.13.21}$$

We consider the condition $P_\lambda\,(I\otimes X_a)P_\kappa\neq 0$ in more detail. Let $V_\lambda$ be the space of the representation $T_\lambda$. We note that $(I\otimes X_a+X_a\otimes I)\cdot V_\lambda\subset V_\lambda$, where $V_\lambda\subset V_\mu\otimes V_\nu$, and, for orthogonal projectors $P_\kappa$ and $P_\lambda$, we deduce

$$P_\lambda\,(I\otimes X_a)P_\kappa=\frac{1}{2}P_\lambda\,(I\otimes X_a-X_a\otimes I)P_\kappa. \tag{3.13.22}$$

One can interpret $(I\otimes X_a-X_a\otimes I)$ as the tensor operator in the adjoint representation and, according to the Wigner–Eckart theorem, the matrix (3.13.22) should be proportional to Clebsch–Gordan coefficients which transform the basis of $V_\lambda$ into the basis of $V_{\rm ad}\otimes V_\kappa$. We note that for existence of the $R$-matrix, it is necessary that the system of equations (3.13.21) have a solution. However, in general, the system (3.13.21) is overdetermined and not always has a solution.

Further we consider the equivalent representations $T_\nu=T_\mu$ and require that the $R$-matrix be symmetric $R_{12}=R_{21}$. Then the space $V_\mu\otimes V_\mu$ is splitted into symmetric $P^{(+)}_{12}\,(V_\mu\otimes V_\mu)$ and antisymmetric $P^{(-)}_{12}\,(V_\mu\otimes V_\mu)$ parts, where $P^{(\pm)}_{12}:=\frac{1}{2}(I\pm P_{12})$. It means that the whole set of projectors (3.13.16) is also divided onto subsets of symmetric and antisymmetric projectors

$$P^{(+)}_\kappa:=P^{(+)}_{12}P_\kappa,\quad P^{(-)}_\sigma:=P^{(-)}_{12}P_\sigma\quad\Rightarrow\quad\sum_\kappa P^{(+)}_\kappa=P^{(+)},\quad\sum_\sigma P^{(-)}_\sigma=P^{(-)}, \tag{3.13.23}$$

and for matrices (3.13.22) we have $P^{(\pm)}_\sigma\,(I\otimes X_a-X_a\otimes I)P^{(\pm)}_\kappa=0$. So, the nonzero contributions to (3.13.22) are $P^{(\pm)}_\sigma\,(I\otimes X_a-X_a\otimes I)P^{(\mp)}_\kappa$. Thus, the representations $T_\lambda$ and $T_\kappa$ in Eqs. (3.13.21) should have the different symmetry, i.e.,

$$V_\lambda\subset P^{(+)}\,V_\mu^{\otimes 2},\quad V_\kappa\subset P^{(-)}\,V_\mu^{\otimes 2},\quad\text{or}\quad V_\lambda\subset P^{(-)}\,V_\mu^{\otimes 2},\quad V_\kappa\subset P^{(+)}\,V_\mu^{\otimes 2},$$

and satisfy conditions

$$T_\lambda\subset{\rm ad}\otimes T_\kappa,\quad T_\kappa\subset{\rm ad}\otimes T_\lambda. \tag{3.13.24}$$

Below, by solving the system of equations (3.13.21), we present $\mathfrak{g}$-invariant $R$-matrices for all simple Lie algebras $\mathfrak{g}$ (except for $\mathfrak{g} = \mathfrak{sl}_N, \mathfrak{e}_8$) in the defining representation $T_\mu = \square \equiv [1]$ [175] (see also [176, 177]).

**1. The $\mathfrak{so}$ and $\mathfrak{sp}$ algebras**

The expansion of the tensor product of defining representations of algebras $\mathfrak{so}(N)$ and $\mathfrak{sp}(N)|_{N=2r}$ is $[1]^{\otimes 2} = [2] + [1^2] + [\emptyset]$, and the highest weights of subrepresentations, Weyl vectors $\delta$ and corresponding eigenvalues $C_{(2)}(\lambda)$, defined in (3.13.20), are

$$\lambda_{[2]} = (2, 0, \dots, 0), \quad \lambda_{[1^2]} = (1, 1, 0, \dots, 0), \quad \lambda_{[\emptyset]} = (0, \dots, 0),$$

for $\mathfrak{so}(N)$:

$$\delta = \sum_{i=1}^{[\frac{N}{2}]} (\frac{N}{2} - i) e^{(i)}, \quad C_{(2)}(\lambda_{[2]}) = 2N, \quad C_{(2)}(\lambda_{[1^2]}) = 2N - 4, \quad C_{(2)}(\lambda_{[\emptyset]}) = 0, \qquad (3.13.25)$$

and for $\mathfrak{sp}(N)$:

$$\delta = \sum_{i=1}^{\frac{N}{2}} (\frac{N}{2} + 1 - i) e^{(i)}, \quad C_{(2)}(\lambda_{[2]}) = 2N + 4, \quad C_{(2)}(\lambda_{[1^2]}) = 2N, \quad C_{(2)}(\lambda_{[\emptyset]}) = 0. \qquad (3.13.26)$$

Below we also need the expansion

$$[1^2] \otimes [2] = [3, 1] + [2, 1^2] + [1^2] + [2]. \qquad (3.13.27)$$

For the $\mathfrak{so}(N)$ case, the representations $[2]$ and $[\emptyset]$ belong to the symmetric part of $[1]^{\otimes 2}$, while the adjoint representation $[1^2] = \mathrm{ad}$ belongs to the antisymmetric part of $[1]^{\otimes 2}$, and we have (cf. (3.13.24), (3.13.27)) $[1^2] \subset [1^2] \otimes [\emptyset]$, $[1^2] \subset [1^2] \otimes [2]$. Therefore, in view of (3.13.25), the system of equations (3.13.21) is written as

$$\frac{\tau_{[\emptyset]}(u)}{\tau_{[1^2]}(u)} = \frac{u - \frac{N}{2} + 1}{u + \frac{N}{2} - 1}, \qquad \frac{\tau_{[2]}(u)}{\tau_{[1^2]}(u)} = \frac{u + 1}{u - 1}, \qquad (3.13.28)$$

and after a renormalization the $\mathfrak{so}$-invariant $R$-matrix (3.13.15) is

$$R(u) = P_{[1^2]} + \frac{\tau_{[2]}(u)}{\tau_{[1^2]}(u)} P_{[2]} + \frac{\tau_{[\emptyset]}(u)}{\tau_{[1^2]}(u)} P_{[\emptyset]} = P_{[1^2]} + \frac{u+1}{u-1} P_{[2]} + \frac{u - \frac{N}{2} + 1}{u + \frac{N}{2} - 1} P_{[\emptyset]}. \qquad (3.13.29)$$

This $R$-matrix is called Zamolodchikov's solution [4, 5] of the Yang–Baxter equation.

For $\mathfrak{sp}(N)$ algebras the representations $[1^2]$ and $[\emptyset]$ belong to the antisymmetric part of $[1]^{\otimes 2}$, while the adjoint representation $[2] = \mathrm{ad}$ belongs to the symmetric part of $[1]^{\otimes 2}$, and we have (cf. (3.13.24), (3.13.27)) $[2] \subset [2] \otimes [\emptyset]$, $[2] \subset [2] \otimes [1^2]$. Therefore, in view of (3.13.26), the system of equations (3.13.21) is written as

$$\frac{\tau_{[\emptyset]}(u)}{\tau_{[2]}(u)} = \frac{u - \frac{N}{2} - 1}{u + \frac{N}{2} + 1}, \qquad \frac{\tau_{[1^2]}(u)}{\tau_{[2]}(u)} = \frac{u - 1}{u + 1}, \qquad (3.13.30)$$

and after a renormalization the $\mathfrak{sp}$-invariant $R$-matrix (3.13.15) is

$$R(u) = P_{[2]} + \frac{\tau_{[1^2]}(u)}{\tau_{[2]}(u)} P_{[1^2]} + \frac{\tau_{[\emptyset]}(u)}{\tau_{[2]}(u)} P_{[\emptyset]} = P_{[2]} + \frac{u-1}{u+1} P_{[1^2]} + \frac{u - \frac{N}{2} - 1}{u + \frac{N}{2} + 1} P_{[\emptyset]}. \qquad (3.13.31)$$

**2. The $\mathfrak{g}_2$ algebra**

Further we denote by $[[n]]$ the $n$-dimensional representation of the Lie algebra $\mathfrak{g}$. The $R$-matrix operator (3.13.15) of the algebra $\mathfrak{g}_2$ in the minimal fundamental representation $[[7]]$ acts in the reducible 49-dimensional space $[[7]]^{\otimes 2}$ which can be expanded in the irreducible components as follows:

$$[[7]] \otimes [[7]] = \mathbb{S}([[7]]^{\otimes 2}) + \mathbb{A}([[7]]^{\otimes 2}) = ([[1]] + [[27]]) + ([[7]] + [[14]]). \tag{3.13.32}$$

Here the fundamental $[[7]]$ and adjoint $[[14]]$ representations of $\mathfrak{g}_2$ are embedded into the antisymmetric $\mathbb{A}$ part of $[[7]]^{\otimes 2}$, while the representations $[[1]]$ and $[[27]]$ compose the symmetric $\mathbb{S}$ part of $[[7]]^{\otimes 2}$. The highest weight vectors $\mu_{[[n]]}$ of the representations $[[n]]$ in the right-hand side of (3.13.32) and Weyl vector $\delta$ are [178] (see also [139] and references therein):

$$\begin{gathered} \mu_{[[1]]} = (0,0,0), \quad \mu_{[[7]]} = \lambda_{(1)} = (0,-1,1), \quad \mu_{[[14]]} = \lambda_{(2)} = (-1,-1,2), \\ \mu_{[[27]]} = 2\lambda_{(1)} = (0,-2,2), \quad \delta = \sum_{i=1}^{2} \lambda_{(i)} = (-1,-2,3), \end{gathered} \tag{3.13.33}$$

where $\lambda_{(1)}$ and $\lambda_{(2)}$ are fundamental weights of $\mathfrak{g}_2$, and we describe the root space of the rank-2 Lie algebra $\mathfrak{g}_2$ as a plane $\mathsf{P}_u$ in 3-dimensional Euclidean space $\mathbb{R}^3$, normal to vector $u = (1,1,1)$. The values (3.13.20) of the quadratic Casimir operators of the representations of the highest weights (3.13.33) are written as

$$C_2^{[[1]]} = 0, \quad C_2^{[[7]]} = 12, \quad C_2^{[[14]]} = 24, \quad C_2^{[[27]]} = 28. \tag{3.13.34}$$

For the case of Lie algebra $\mathfrak{g}_2$ the conditions (3.13.24) are (see, e.g., [179] about tensor product of $\mathfrak{g}_2$ representations):

$$[[1]] \subset \mathrm{ad} \otimes [[14]], \quad [[27]] \subset \mathrm{ad} \otimes [[7]], \quad [[27]] \subset \mathrm{ad} \otimes [[14]],$$

where $\mathrm{ad} = [[14]]$, and we write the system of equations (3.13.21) as

$$\frac{\tau_{[[1]]}(u)}{\tau_{[[14]]}(u)} = \frac{u-6}{u+6}, \qquad \frac{\tau_{[[7]]}(u)}{\tau_{[[27]]}(u)} = \frac{u-4}{u+4}, \qquad \frac{\tau_{[[14]]}(u)}{\tau_{[[27]]}(u)} = \frac{u-1}{u+1}. \tag{3.13.35}$$

Finally, after a normalization, the $\mathfrak{g}_2$-invariant $R$-matrix (3.13.15) in the fundamental representation $[[7]]$ acquires the form [175] (see also [176, 177] with $u \to -u$):

$$\begin{aligned} R(u) &= \frac{\tau_{[[1]]}(u)}{\tau_{[[27]]}(u)} P_{[[1]]} + \frac{\tau_{[[7]]}(u)}{\tau_{[[27]]}(u)} P_{[[7]]} + \frac{\tau_{[[14]]}(u)}{\tau_{[[27]]}(u)} P_{[[14]]} + P_{[[27]]} = \\ &= \frac{(u-6)(u-1)}{(u+6)(u+1)} P_{[[1]]} + \frac{u-4}{u+4} P_{[[7]]} + \frac{u-1}{u+1} P_{[[14]]} + P_{[[27]]}. \end{aligned} \tag{3.13.36}$$

**3. The $\mathfrak{f}_4$ algebra**

The $\mathfrak{f}_4$-invariant $R$-matrix (3.13.15) in the minimal fundamental representation $[[26]]$ acts in the reducible 676-dimensional space $[[26]]^{\otimes 2}$ which is expanded in the irreducible components as follows:

$$[[26]]^{\otimes 2} = \mathbb{S}([[26]]^{\otimes 2}) + \mathbb{A}([[26]]^{\otimes 2}) = ([[1]] + [[26]] + [[324]]) + ([[52]] + [[273]]), \tag{3.13.37}$$

where representations $[[1]], [[26]]$, and $[[324]]$ belong to the symmetric part of $[[26]]^{\otimes 2}$, while the adjoint representation $[[52]]$ and representation $[[273]]$ belong to the antisymmetric part of $[[26]]^{\otimes 2}$.

The highest weight vectors of the representations in (3.13.37) and $\mathfrak{f}_4$ Weyl vector $\delta$ are [178] (see also [139] and references therein):

$$\begin{gathered} \mu_{[[1]]} = (0,0,0,0), \quad \mu_{[[26]]} = \lambda_{(4)} = (0,0,0,1), \quad \mu_{[[52]]} = \lambda_{(1)} = (1,0,0,1), \\ \mu_{[[273]]} = \lambda_{(3)} = \tfrac{1}{2}(1,1,1,3), \quad \mu_{[[324]]} = 2\lambda_{(4)} = (0,0,0,2), \quad \delta = \sum_{i=1}^{4} \lambda_{(i)} = \tfrac{1}{2}(5,3,1,11), \end{gathered} \tag{3.13.38}$$

where $\lambda_{(i)}|_{i=1,\dots,4}$ are fundamental weights of $\mathfrak{f}_4$. Now we deduce the values (3.13.20) of the quadratic Casimir operators, which correspond to the highest weights (3.13.33)

$$C_2^{[[1]]} = 0, \quad C_2^{[[26]]} = 12, \quad C_2^{[[52]]} = 18, \quad C_2^{[[273]]} = 24, \quad C_2^{[[324]]} = 26. \tag{3.13.39}$$

The analogs of the conditions (3.13.24) for the Lie algebra $\mathfrak{f}_4$ have the form

$$[[52]] \subset \text{ad} \otimes [[1]], \quad [[324]] \subset \text{ad} \otimes [[52]], \quad [[324]] \subset \text{ad} \otimes [[273]], \quad [[273]] \subset \text{ad} \otimes [[26]],$$

where $\text{ad} \equiv [[52]]$, and the system of equations (3.13.21) is represented as

$$\frac{\tau_{[[1]]}(u)}{\tau_{[[52]]}(u)} = \frac{u - \frac{9}{2}}{u + \frac{9}{2}}, \quad \frac{\tau_{[[52]]}(u)}{\tau_{[[324]]}(u)} = \frac{u-2}{u+2}, \quad \frac{\tau_{[[273]]}(u)}{\tau_{[[324]]}(u)} = \frac{u - \frac{1}{2}}{u + \frac{1}{2}}, \quad \frac{\tau_{[[26]]}(u)}{\tau_{[[273]]}(u)} = \frac{u-3}{u+3}. \tag{3.13.40}$$

Finally, after a normalization, the $\mathfrak{f}_4$-invariant $R$-matrix in the fundamental representation $[[26]]$ has the form [175] (see also [176, 177] with $u \to -2u$):

$$\begin{aligned} R(u) &= \tfrac{\tau_{[[1]]}(u)}{\tau_{[[324]]}(u)} P_{[[1]]} + \tfrac{\tau_{[[26]]}(u)}{\tau_{[[324]]}(u)} P_{[[26]]} + \tfrac{\tau_{[[52]]}(u)}{\tau_{[[324]]}(u)} P_{[[52]]} + \tfrac{\tau_{[[273]]}(u)}{\tau_{[[324]]}(u)} P_{[[273]]} + P_{[[324]]} = \\ &= \tfrac{(u-9/2)(u-2)}{(u+9/2)(u+2)} P_{[[1]]} + \tfrac{(u-3)(u-1/2)}{(u+3)(u+1/2)} P_{[[26]]} + \tfrac{u-2}{u+2} P_{[[52]]} - \tfrac{u-1/2}{u+1/2} P_{[[273]]} + P_{[[324]]}. \end{aligned} \tag{3.13.41}$$

## 4. The $\mathfrak{e}_6$ algebra

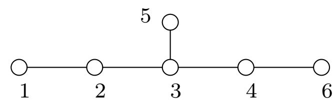
The 78-dimensional algebra $\mathfrak{e}_6$ has two inequivalent minimal fundamental representations $[[27]]$ and $[[\overline{27}]]$. Here we consider $\mathfrak{e}_6$-invariant $R$-matrix (3.13.15) which acts in the space of reducible representation

$$[[27]]^{\otimes 2} = \mathbb{S}([[27]]^{\otimes 2}) + \mathbb{A}([[27]]^{\otimes 2}) = ([[\overline{27}]] + [[351]]_1) \; + \; ([[351]]_2), \qquad \text{ad} \equiv [[78]]\,. \tag{3.13.42}$$

The highest weight vectors $\mu_{[[n]]}$ of the representations in (3.13.42) and Weyl vector $\delta$ for $\mathfrak{e}_6$ are [139, 178]

$$\begin{gathered} \mu_{[[27]]} = \lambda_{(1)} = (-\tfrac{1}{3}, -\tfrac{1}{3}, 1, 0, 0, 0, 0, \tfrac{1}{3}), \quad \mu_{[[\overline{27}]]} = \lambda_{(6)} = \tfrac{2}{3}(-1, -1, 0, 0, 0, 0, 0, 1), \\ \mu_{[[351]]_1} = 2\lambda_{(1)} = (-\tfrac{2}{3}, -\tfrac{2}{3}, 2, 0, 0, 0, 0, \tfrac{2}{3}), \quad \mu_{[[351]]_2} = \lambda_{(2)} = (-\tfrac{2}{3}, -\tfrac{2}{3}, 1, 1, 0, 0, 0, \tfrac{2}{3}), \\ \delta = \sum_{i=1}^{6} \lambda_{(i)} = (-4, -4, 4, 3, 2, 1, 0, 4), \end{gathered} \tag{3.13.43}$$

where $\lambda_{(i)}|_{i=1,\dots,6}$ are fundamental weights of $\mathfrak{e}_6$, and we numerate nodes of $\mathfrak{e}_6$ Dynkin diagram as follows:




The values (3.13.20) of the quadratic Casimir operators, which correspond to the representations with highest weights (3.13.43), are

$$C_2^{[[\overline{27}]]} = \tfrac{52}{3}, \quad C_2^{[[351]]_1} = \tfrac{112}{3}, \quad C_2^{[[351]]_2} = \tfrac{100}{3}. \tag{3.13.44}$$

We note that, in view of the symmetry of the $\mathfrak{e}_6$ Dynkin diagram, the values (3.13.44) are invariant under the change of the fundamental weights in (3.13.43): $\lambda_{(1)} \leftrightarrow \lambda_{(6)}$ and $\lambda_{(2)} \leftrightarrow \lambda_{(4)}$. This symmetry also means that the $R$-matrix in the representation $[[\overline{27}]]^{\otimes 2}$ has the same form as the $R$-matrix in the representation $[[27]]^{\otimes 2}$. The analogs of the conditions (3.13.24) for the Lie algebra $\mathfrak{e}_6$ are (see, e.g., [179])

$$[[351]]_2 \subset \mathrm{ad} \otimes [[\overline{27}]], \quad [[351]]_1 \subset \mathrm{ad} \otimes [[351]]_2,$$

where $\mathrm{ad} \equiv [[78]]$, and the system of equations (3.13.21) is written as

$$\frac{\tau_{[[\overline{27}]]}(u)}{\tau_{[[351]]_2}(u)} = \frac{u-4}{u+4}, \qquad \frac{\tau_{[[351]]_1}(u)}{\tau_{[[351]]_2}(u)} = \frac{u+1}{u-1}\,. \tag{3.13.45}$$

Finally, the $\mathfrak{e}_6$-invariant $R$-matrix in the representation [[27]] acquires the form [175–177]

$$\begin{array}{c} R(u) = \frac{\tau_{[[\overline{27}]]}(u)}{\tau_{[[351]]_2}(u)} P_{[[\overline{27}]]} + \frac{\tau_{[[351]]_1}(u)}{\tau_{[[351]]_2}(u)} P_{[[351]]_1} + P_{[[351]]_2} = \\ = \frac{u-4}{u+4} P_{[[\overline{27}]]} + \frac{u+1}{u-1} P_{[[351]]_1} + P_{[[351]]_2}. \end{array} \tag{3.13.46}$$

**5. The $\mathfrak{e}_7$ algebra**

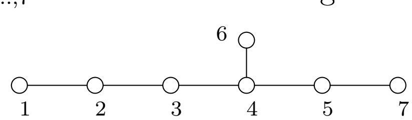
Here we consider $\mathfrak{e}_7$-invariant $R$-matrix (3.13.15) which acts in the space of the representation

$$[[56]]^{\otimes 2} = \mathbb{S}([[56]]^{\otimes 2}) + \mathbb{A}([[56]]^{\otimes 2}) = ([[133]] + [[1463]]) + ([[1]] + [[1539]])\,, \tag{3.13.47}$$

where $[[133]] \equiv \mathrm{ad}$ is the adjoint representation of $\mathfrak{e}_7$. The highest weight vectors $\mu_{[[n]]}$ of the representations in (3.13.47) and Weyl vector $\delta$ for the algebra $\mathfrak{e}_7$ are [139, 178]

$$\begin{array}{c} \mu_{[[1]]} = (0,0,0,0,0,0,0,0), \quad \mu_{[[133]]} = \lambda_{(7)} = (-1,0,0,0,0,0,0,1), \\ \mu_{[[1463]]} = 2\lambda_{(1)} = (-1,2,0,0,0,0,0,1), \quad \mu_{[[1539]]} = \lambda_{(2)} = (-1,1,1,0,0,0,0,1), \\ \mu_{[[56]]} = \lambda_{(1)} = (-\frac{1}{2},1,0,0,0,0,0,\frac{1}{2}), \quad \delta = \sum\limits_{i=1}^{7} \lambda_{(i)} = (-\frac{17}{2},5,4,3,2,1,0,\frac{17}{2}), \end{array} \tag{3.13.48}$$

where $\lambda_{(i)}|_{i=1,\dots,7}$ are fundamental weights of $\mathfrak{e}_7$ and we numerate nodes in Dynkin diagram as follows:




The values (3.13.20) of the quadratic Casimir operators, which correspond to the representations with highest weights (3.13.48), are

$$C_2^{[[56]]} = \tfrac{57}{2}, \quad C_2^{[[133]]} = 36, \quad C_2^{[[1463]]} = 60, \quad C_2^{[[1539]]} = 56. \tag{3.13.49}$$

The analogs of the conditions (3.13.24) for the Lie algebra $\mathfrak{e}_7$ have the form (see, e.g., [179])

$$[[133]] \subset \mathrm{ad} \otimes [[1]], \quad [[1539]] \subset \mathrm{ad} \otimes [[1463]], \quad [[1539]] \subset \mathrm{ad} \otimes [[133]],$$

where ad $\equiv [[133]]$, and the system of equations (3.13.21) is written as

$$\frac{\tau_{[[1]]}(u)}{\tau_{[[133]]}(u)} = \frac{u-9}{u+9}, \qquad \frac{\tau_{[[133]]}(u)}{\tau_{[[1539]]}(u)} = \frac{u-5}{u+5}, \qquad \frac{\tau_{[[1463]]}(u)}{\tau_{[[1539]]}(u)} = \frac{u+1}{u-1}\,. \tag{3.13.50}$$

Thus, the $\mathfrak{e}_7$-invariant solution (3.13.15) of the Yang–Baxter equation in the defining representation [[56]] has the form

$$R(u) = \frac{(u-9)(u-5)}{(u+9)(u+5)} P_{[[1]]} + \frac{u-5}{u+5} P_{[[133]]} + \frac{u+1}{u-1} P_{[[1463]]} + P_{[[1539]]}. \tag{3.13.51}$$

**6. The $\mathfrak{e}_8$ algebra**

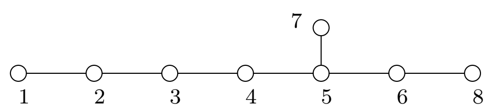
For the exceptional Lie algebra $\mathfrak{e}_8$ the adjoint and minimal fundamental representations coincide and have dimension 248. The tensor product of its two 248-dimensional representations has the following decomposition into irreducible representations [179, 180]:

$$[[248]]^{\otimes 2} = \mathbb{S}([[248]]^{\otimes 2}) + \mathbb{A}([[248]]^{\otimes 2}) = ([[1]] + [[3875]] + [[27000]]) + ([[248]] + [[30380]])\ . \tag{3.13.52}$$

The highest weight vectors $\mu_{[[n]]}$ of the representations in (3.13.52) and Weyl vector $\delta$ for the algebra $\mathfrak{e}_8$ are [139, 178]

$$\begin{gathered} \mu_{[[248]]} = \lambda_{(1)} = (1,0,0,0,0,0,0,1), \quad \mu_{[[3875]]} = \lambda_{(8)} = (0,0,0,0,0,0,0,2), \\ \mu_{[[27000]]} = 2\lambda_{(1)} = (2,0,0,0,0,0,0,2), \quad \mu_{[[30380]]} = \lambda_{(2)} = (1,1,0,0,0,0,0,2), \\ \delta = \sum_{i=1}^{8} \lambda_{(i)} = (6,5,4,3,2,1,0,23), \end{gathered} \tag{3.13.53}$$

where $\lambda_{(i)}|_{i=1,\dots,8}$ are fundamental weights of $\mathfrak{e}_8$ and we numerate nodes in Dynkin diagram as follows:




The values (3.13.20) of the quadratic Casimir operators, which correspond to the representations with highest weights (3.13.53), are

$$C_2^{[[248]]} = 60, \quad C_2^{[[3875]]} = 96, \quad C_2^{[[27000]]} = 124, \quad C_2^{[[30380]]} = 120. \tag{3.13.54}$$

The analogs of the conditions (3.13.24) for the Lie algebra $\mathfrak{e}_8$ have the form (see, e.g., [179])

$$\begin{gathered} [[248]] \subset \mathrm{ad} \otimes [[1]], \quad [[3875]] \subset \mathrm{ad} \otimes [[248]], \quad [[27000]] \subset \mathrm{ad} \otimes [[248]], \\ [[30380]] \subset \mathrm{ad} \otimes [[27000]], \quad [[30380]] \subset \mathrm{ad} \otimes [[3875]], \end{gathered}$$

where ad $\equiv [[248]]$, and the part of the system of equations (3.13.21) is written as

$$\frac{\tau_{[[248]]}(u)}{\tau_{[[3875]]}(u)} = \frac{u-9}{u+9}, \quad \frac{\tau_{[[248]]}(u)}{\tau_{[[27000]]}(u)} = \frac{u-16}{u+16}, \quad \frac{\tau_{[[27000]]}(u)}{\tau_{[[30380]]}(u)} = \frac{u+1}{u-1}, \quad \frac{\tau_{[[30380]]}(u)}{\tau_{[[3875]]}(u)} = \frac{u+6}{u-6}\,. \tag{3.13.55}$$

It is clear that Eqs. (3.13.55) are inconsistent. Thus, the $\mathfrak{e}_8$-invariant $R$-matrix in the minimal fundamental (adjoint) representation [[248]] does not exist. This fact is in agreement with the general statement [143, 174] that the adjoint representation of $\mathfrak{e}_8$ can not be lifted to the representation of the Yangian $Y(\mathfrak{e}_8)$.

In papers [176, 177], the $R$-matrix solutions (3.13.29), (3.13.31), (3.13.36), (3.13.41), (3.13.46), and (3.13.51) are written as rational functions of the split Casimir operators (3.13.3) in the defining representations. In particular, for Lie algebras $\mathfrak{g} = \mathfrak{so}, \mathfrak{sp}, \mathfrak{e}_6$ we have [176, 177]

$$R(u) = \frac{u + \frac{1}{2}(\hat{C} + a_{\mathfrak{g}})}{u - \frac{1}{2}(\hat{C} + a_{\mathfrak{g}})}, \tag{3.13.56}$$

where $a_{\mathfrak{so}} = 1$, $a_{\mathfrak{sp}} = -1$, $a_{\mathfrak{e}_6} = 2/3$ and the split Casimir operators $\hat{C}$ are normalized such that $\hat{C}^{ki}_{jk} = C_2(\lambda)\delta^i_j$ (here $\lambda$ is a highest weight of the defining representation of $\mathfrak{g}$ and the value $C_2(\lambda)$ is given by (3.13.20)).

### *3.13.2. Yangians $Y(\mathfrak{g})$*

The Yangians $Y(\mathfrak{g})$ can be defined by means of the rational $R$-matrix solutions (3.13.10), (3.13.15) of the Yang–Baxter equations (3.13.8). First, we consider equations (cf. (3.9.13)

$$R_{12}(u-v)\, L_1(u)\, L_2(v) = L_2(v)\, L_1(u)\, R_{12}(u-v), \tag{3.13.57}$$

where $u, v$ are spectral parameters, indices $1, 2$ numerate the spaces $V$ of the defining representation $T$ in the product $V \otimes V$, and $R_{12}(u)$ is the $\mathfrak{g}$-invariant $R$-matrix in the representation $T \otimes T$. We search the elements $L^i_{\ j}(u)$ of the quadratic algebra (3.13.57) in the form (cf. (3.9.17))

$$L^i_{\ j}(u) = \delta^i_j + \sum_{k=1}^{\infty} T^{(k)}{}^i_{\ j} u^{-k} = \delta^i_j + \frac{1}{u}\mathbf{I}^i_j + \frac{1}{u^2}\Big(\frac{1}{2}(\mathbf{I}^2)^i_{\ j} + \mathbf{J}^i_{\ j}\Big) + \frac{1}{u^3}\dots, \tag{3.13.58}$$

where $T^{(k)}{}^i_{\ j}$ $(k > 1)$ are the generators of the Yangian $Y(\mathfrak{g})$ [10, 143] and we introduce the notation

$$T^{(1)}{}^i_{\ j} \equiv \mathbf{I}^i_j = I^a (T_a)^i_{\ j}, \quad T^{(2)}{}^i_{\ j} \equiv \frac{1}{2}(\mathbf{I}^2)^i_{\ j} + \mathbf{J}^i_{\ j}, \quad \mathbf{J}^i_{\ j} := J^a (T_a)^i_{\ j}.$$

Here $(T_a)^i_{\ j} = T^i_j(X_a)$ are generators of $\mathfrak{g}$ in the representation $T$. Now we substitute expansions (3.13.10), (3.13.11), and (3.13.58) into (3.13.57), multiply both sides by $(u-v)^2$ and consider Eq. (3.13.57) in the limit $u, v \to \infty$. We take into account identities (cf. (3.13.14))

$$(\mathbf{I}_1 + \mathbf{I}_2)\hat{C}_{12} = \hat{C}_{12}(\mathbf{I}_1 + \mathbf{I}_2), \quad (\mathbf{J}_1 + \mathbf{J}_2)\hat{C}_{12} = \hat{C}_{12}(\mathbf{J}_1 + \mathbf{J}_2),$$

where $\hat{C}_{12}$ is the split Casimir operator in the defining representation $T$. Then the terms of zero order in $u, v$ of Eqs. (3.13.57) give relations (3.13.1):

$$[\mathbf{I}_1,\ \mathbf{I}_2] = \hat{C}_{12}\mathbf{I}_1 - \mathbf{I}_1\hat{C}_{12} \qquad \Rightarrow \qquad [I_a,\ I_b] = X^d_{ab}\, I_d, \tag{3.13.59}$$

which means that coefficients $I_a$ are the basis elements of the Lie algebra $\mathfrak{g}$. The terms of order $u^{-2}v$ and $uv^{-2}$ of Eqs. (3.13.57) give commutation relations

$$[\mathbf{I}_1,\ \mathbf{J}_2] = \hat{C}_{12}\mathbf{J}_1 - \mathbf{J}_1\hat{C}_{12} \ \equiv\ \mathbf{J}_2\hat{C}_{12} - \hat{C}_{12}\mathbf{J}_2 \qquad \Rightarrow \qquad [I_a,\ J_b] = J_d\, X^d_{ab}, \tag{3.13.60}$$

which means that the elements $J_b$ form the adjoint representation of $\mathfrak{g}$. We also note useful generalizations of (3.13.59) and (3.13.60):

$$[\mathbf{I}_1,\ \mathbf{I}_2^k] = \mathbf{I}_2^k\,\hat{C}_{12} - \hat{C}_{12}\,\mathbf{I}_2^k, \qquad [\mathbf{I}_1,\ \mathbf{J}_2^k] = \mathbf{J}_2^k\,\hat{C}_{12} - \hat{C}_{12}\,\mathbf{J}_2^k. \tag{3.13.61}$$

The commutator $[\mathbf{J}_1,\ \mathbf{J}_2]$ is not fully specified by Eq. (3.13.57) if we know the expansion of $R$-matrix (3.13.10) only up to the order $u^{-2}$.

We define the Yangian $Y(\mathfrak{g})$ as the enveloping algebra generated by the basis elements $I_a|_{a=1,\dots,\dim\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$, the additional set of elements $J_a|_{a=1,\dots,\dim\mathfrak{g}}$, which form the adjoint representation (3.13.60) of $\mathfrak{g}$ and with a nontrivial noncommutative coproduct $\Delta$ : $Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ which is defined by

$$\Delta(L^i_{\ j}(u)) = L^i_{\ k}(u) \otimes L^k_{\ j}(u). \tag{3.13.62}$$

The substitution of (3.13.58) into (3.13.62) gives

$$\begin{gathered}
\Delta(\mathbf{I}^i_{\ j}) = \mathbf{I}^i_{\ j} \otimes 1 + 1 \otimes \mathbf{I}^i_{\ j} \quad \Rightarrow \quad \Delta(I_a) = I_a \otimes 1 + 1 \otimes I_a, \\
\Delta(\mathbf{J}^i_{\ j}) = \mathbf{J}^i_{\ j} \otimes 1 + 1 \otimes \mathbf{J}^i_{\ j} + \frac{1}{2}\big(\mathbf{I}^i_{\ k} \otimes \mathbf{I}^k_{\ j} - \mathbf{I}^k_{\ j} \otimes \mathbf{I}^i_{\ k}\big) \quad \Rightarrow \\
\Delta(J_a) = J_a \otimes 1 + 1 \otimes J_a + \frac{1}{2} X_{abc}\, I^b \otimes I^c,
\end{gathered} \tag{3.13.63}$$

where $X_{abc} = X^d_{ab}\, \mathsf{g}_{dc}$. Finally, we remark that the commutator $[J_a,\ J_b]$ is constrained by the requirement that $\Delta$ be a homomorphism. Indeed, we have [143, 173]:

$$\begin{gathered}
[J_a,\ [J_b,\ I_c]] - [I_a,\ [J_b,\ J_c]] = a_{abc}{}^{efg}\, \{I_e, I_f, I_g\}, \\
[[J_a,\ J_b],\ [I_c,\ J_d]] + [[J_c,\ J_d],\ [I_a,\ J_b]] = (a_{abh}{}^{efg}\, X_{hcd} + a_{cdh}{}^{efg}\, X_{hab})\{I_e, I_f, J_g\},
\end{gathered}$$

where $a_{abh}{}^{efg} = \frac{1}{24} X^e_{ia} X^f_{jb} X^g_{kh} X^{ijk}$ and $\{x_1, x_2, x_3\} = \sum\limits_{e \neq f \neq g} x_e, x_f, x_g$.

### *3.14. Quantum Knizhnik–Zamolodchikov equations*

In Subsections **3.8** and **3.12**, by using $R$-matrix representations for the Hecke and Birman–Murakami–Wenzl algebras, we have found the trigonometric solutions $R(x)$ of the Yang–Baxter equations (Baxterized $R$-matrices). In this subsection, we show that, for every trigonometric solution $R(x)$ of the Yang–Baxter equations (3.8.2), one can construct the set of difference equations which are called *quantum Knizhnik–Zamolodchikov equations*. These equations are important, since their solutions are related (see, e.g., [181–183] and references therein) to the correlation functions in spin chain models associated with the same trigonometric matrix $R(x)$.

In this subsection, we follow the presentation of the papers [184–186].

Consider a tensor function $\Psi^{1\dots N\rangle}(z_1, \dots, z_N) \in V^{\otimes N}$ ($z_i \in \mathbb{C}$, $i = 1, \dots, N$) which satisfies a system of difference equations

$$T_{(i)}\, \Psi^{1\dots N\rangle}(z_1, \dots, z_N) = A^{(i)}_{1\dots N}(z_1, \dots, z_N)\, \Psi^{1\dots N\rangle}(z_1, \dots, z_N), \tag{3.14.1}$$

where operator $T_{(i)}$ is defined as

$$T_{(i)}\, \Psi^{1\dots N\rangle}(z_1, \dots, z_N) := \Psi^{1\dots N\rangle}(z_1, \dots, z_{i-1}, p z_i, z_{i+1}, \dots, z_N), \tag{3.14.2}$$

$A^{(i)}_{1\dots N}(z_1,\dots,z_N) \in \mathrm{End}(V^{\otimes N})$ is called discrete connection and indices $1,\dots,N$ denote the numbers of the vector spaces $V$ in $V^{\otimes N}$. A consistence condition $T_{(i)}T_{(j)} = T_{(j)}T_{(i)}$ of the system (3.14.1) requires additional constraints on the discrete connection $A^{(j)}_{1\dots N}(z_1\dots z_N)$:

$$T_{(i)}A^{(j)}_{1\dots N}T^{-1}_{(i)}A^{(i)}_{1\dots N} = T_{(j)}A^{(i)}_{1\dots N}T^{-1}_{(j)}A^{(j)}_{1\dots N} \quad \Rightarrow \tag{3.14.3}$$

$$\mathsf{A}^{(j)}_{1\dots N}\,\mathsf{A}^{(i)}_{1\dots N} = \mathsf{A}^{(i)}_{1\dots N}\,\mathsf{A}^{(j)}_{1\dots N}, \qquad \mathsf{A}^{(i)} := T^{-1}_{(i)}A^{(i)}_{1\dots N}. \tag{3.14.4}$$

Connections $A^{(i)}_{1\dots N}$ and $\mathsf{A}^{(j)}_{1\dots N}$ which satisfy (3.14.3) and (3.14.4) are called flat (or integrable).

Now we introduce the following discrete connection [184]:

$$\begin{gathered} A^{(j)}_{1\dots N}(z_1,\dots,z_N) = T_{(j)}R_{j\,j-1}\dots R_{j\,2}\,R_{j\,1}T^{-1}_{(j)}\,D_j\,R^{-1}_{N\,j}R^{-1}_{N-1\,j}\dots R^{-1}_{j+1\,j} = \\ = R_{j\,j-1}(\tfrac{z_j}{z_{j-1}}p)\dots R_{j\,2}(\tfrac{z_j}{z_2}p)\,R_{j\,1}(\tfrac{z_j}{z_1}\,p)\,D_j\,R^{-1}_{N\,j}(\tfrac{z_N}{z_j})R^{-1}_{N-1\,j}(\tfrac{z_{N-1}}{z_j})\dots R^{-1}_{j+1\,j}(\tfrac{z_{j+1}}{z_j}), \end{gathered} \tag{3.14.5}$$

where $R_{i\,j} := R_{i\,j}(z_i/z_j)$ is the $R$-matrix which acts nontrivially only in the vector spaces $V$ with numbers $i,j$ in $V^{\otimes N}$ and satisfies the Yang–Baxter equation (3.8.2) in the form ($R_{ij}(x) = P_{ij}\hat{R}_{ij}(x)$):

$$R_{ij}(x)\,R_{ik}(xy)\,R_{jk}(y) = R_{jk}(y)\,R_{ik}(xy)\,R_{ij}(x). \tag{3.14.6}$$

The unitarity condition $R_{ij}(x)R_{ji}(x^{-1}) = \mathbf{1}$ is also required (cf. (3.8.8), (3.12.19)) for these $R$-matrices. The constant matrix $D_i$ acts in the $i$th vector space $V_i$ and obeys $R_{ij}D_iD_j = D_iD_jR_{ij}$. Equations (3.14.1) with discrete connection (3.14.5) are called quantum Knizhnik–Zamolodchikov (q-KZ) equations. It is convenient to rewrite the definition of the discrete connection (3.14.5) in the form of commutative matrices (3.14.4) as follows:

$$\begin{gathered} \mathsf{A}^{(j)}_{1\dots N}(z_1,\dots,z_N) = T^{-1}_{(j)}\,A^{(j)}_{1\dots N}(z_1,\dots,z_N) \;=\; \hat{\hat{R}}_{j-1}\dots,\hat{\hat{R}}_1\,\overline{\mathbf{X}}\,\hat{\hat{R}}^{-1}_{N-1}\hat{\hat{R}}^{-1}_{N-2}\dots\hat{\hat{R}}^{-1}_j, \\ \overline{\mathbf{X}} \;:=\; T^{-1}_{(1)}\,D_1\,\mathbf{P}_{1,2}\mathbf{P}_{2,3}\dots\mathbf{P}_{N-1,N}, \end{gathered} \tag{3.14.7}$$

where $\mathbf{P}_{j,k} = P_{j,k}\cdot P_{z_j,z_k}$ and $P_{z_j,z_k}$ is an operator which permutes the spectral parameters $z_j$ and $z_k$:

$$P_{z_j,z_k}\cdot f(z_1,\dots,z_k,\dots,z_j,\dots,z_N) = f(z_1,\dots,z_j,\dots,z_k,\dots,z_N)\cdot P_{z_j,z_k},$$

such that $\hat{\hat{R}}_j := \mathbf{P}_{j,j+1}R_{j,j+1}(z_j/z_{j+1})$ realize generators of the braid group $\mathcal{B}_N$ (see Eqs. (4.1.1) in Subsection **4.1**). We note that operator $\overline{\mathbf{X}}$ satisfies the relations

$$\hat{\hat{R}}_{k+1}\,\overline{\mathbf{X}} = \overline{\mathbf{X}}\,\hat{\hat{R}}_k \quad (k=1,\dots,N-2), \qquad \hat{\hat{R}}_1\,\overline{\mathbf{X}}^2 = \overline{\mathbf{X}}^2\,\hat{\hat{R}}_{N-1}, \tag{3.14.8}$$

and it can be considered as the image of an additional element which extends the group $\mathcal{B}_N$.

**Proposition 3.15** *(see [184, 185]). Discrete connection (3.14.5) is the flat discrete connection (i.e., satisfies (3.14.3)), and therefore the system of equations (3.14.1) with connection (3.14.5) is consistent.*

**Proof.** Indeed, we have from (3.14.3) for $j > i$:

$$\begin{gathered} T_{(i)}T_{(j)}R_{j\,j-1}\dots R_{j\,1}T^{-1}_{(j)}D_jR^{-1}_{Nj}\dots R^{-1}_{j+1\,j}R_{i\,i-1}\dots R_{i\,1}T^{-1}_{(i)}D_iR^{-1}_{Ni}\dots R^{-1}_{i+1\,i} = \\ = T_{(i)}R_{i\,i-1}\dots R_{i\,1}T^{-1}_{(i)}D_iT_{(j)}R^{-1}_{Ni}\dots R^{-1}_{i+1\,i}R_{j\,j-1}\dots R_{j\,1}T^{-1}_{(j)}D_jR^{-1}_{Nj}\dots R^{-1}_{j+1\,j}. \end{gathered}$$

In the left-hand side, we obtain for $j > i$:

$$T_{(i)}T_{(j)}R_{j\,j-1}\dots R_{j\,1}R_{i\,i-1}\dots R_{i\,1}T^{-1}_{(i)}T^{-1}_{(j)}D_iD_jR^{-1}_{N\,j}\dots R^{-1}_{j+1\,j}R^{-1}_{Ni}\dots R^{-1}_{i+1\,i}. \tag{3.14.9}$$

Then we use here identities for transfer-matrices

$$R_{j\,j-1}\dots R_{j\,1}(R_{i\,i-1}\dots R_{i\,1}) = (R_{i\,i-1}\dots R_{i\,1})R_{j\,j-1}\dots R_{j\,i+1}R_{j\,i-1}\dots R_{j\,1}R_{ji},$$

$$(R^{-1}_{Nj}\dots R^{-1}_{j+1\,j})R^{-1}_{N\,i}\dots R^{-1}_{i+1\,i} = R^{-1}_{ji}R^{-1}_{Ni}\dots R^{-1}_{j+1\,i}R^{-1}_{j-1\,i}\dots R^{-1}_{i+1\,i}(R^{-1}_{Nj}\dots R^{-1}_{j+1\,j})$$

and obvious relations $[T^{-1}_{(i)}T^{-1}_{(j)},\, R_{i\,j}] = 0 = [D_iD_j,\, R_{i\,j}]$. As a result, we obtain for the l.h.s. (3.14.9):

$$\begin{gathered} T_{(i)}T_{(j)}(R_{i\,i-1}\dots R_{i\,1})R_{j\,j-1}\dots R_{j\,i+1}R_{j\,i-1}\dots R_{j\,1}T^{-1}_{(i)}T^{-1}_{(j)}D_iD_j \times \\ \times R^{-1}_{Ni}\dots R^{-1}_{j+1\,i}R^{-1}_{j-1\,i}\dots R^{-1}_{i+1\,i}(R^{-1}_{Nj}\dots R^{-1}_{j+1\,j}). \end{gathered} \tag{3.14.10}$$

In the r.h.s., we use $[R_{i\,j}D_k] = 0 = [R_{i\,j}T_k]$ for $i, j \neq k$ and the identity

$$R^{-1}_{Ni}\cdots R^{-1}_{i+1\,i}R_{j\,j-1}\cdots R_{j\,1} = R_{j\,j-1}\cdots R_{j\,i+1}R_{j\,i-1}\cdots R_{j\,1}R^{-1}_{Ni}\cdots R^{-1}_{j+1\,i}R^{-1}_{j-1\,i}\cdots R^{-1}_{i+1\,i},$$

which gives for the r.h.s. just the same answer (3.14.10) as for the l.h.s. ∎

At the end of this subsection, we present the definition of the q-KZ equations due to F. Smirnov [186]. Define the $R$-matrix and operator $D_i$ as follows:

$$\begin{gathered} \Psi^{1\dots N\rangle}(z_1,\dots,z_{i+1},z_i,\dots z_N) = \hat{R}_{i\,i+1}(z_i/z_{i+1})\Psi^{1\dots N\rangle}(z_1,\dots,z_i,z_{i+1},\dots,z_N), \\ \Psi^{1\dots N\rangle}(p\,z_1,z_2,\dots,z_N) = D_1\,\Psi^{2\dots N,1\rangle}(z_2,z_3,\dots,z_N,z_1). \end{gathered} \tag{3.14.11}$$

One can explicitly show that Eqs. (3.14.11) lead to Eqs. (3.14.1), (3.14.2), (3.14.5). Indeed, one can cyclically permute spectral parameters in $\Psi^{1\dots N\rangle}$ by means of the first equation in (3.14.11) and then use the second equation in (3.14.11). The self-consistence of Eqs. (3.14.11) can be checked directly. It also follows from the self-consistence of the extended Zamolodchikov algebra with generators $\{A_i(z_i), Q\}$ $(i = 1,\dots,N)$:

$$\hat{R}_{1\,2}(z_1/z_2)\,A^{1\rangle}(z_1)\,A^{2\rangle}(z_2) = A^{1\rangle}(z_2)\,A^{2\rangle}(z_1), \quad D_1A^{1\rangle}(z_1)\,Q = Q\,A^{1\rangle}(p\,z_1),$$

and remark that Eqs. (3.14.11) can be formally produced from the representation

$$\Psi^{1\dots N\rangle}(z_1,\dots,z_N) = \mathrm{Tr}\left(QA^{1\rangle}(z_1)\,A^{2\rangle}(z_2)\dots A^{N\rangle}(z_N)\right).$$

The semiclassical limit of the q-KZ equations (if we take the the trigonometric $R$-matrices (3.8.5) and (3.12.14) and consider their Yangian limits) gives [184] the usual Knizhnik–Zamolodchikov equations. Moreover, the flat connections (3.14.5) (and their semiclassical limits) are related to Dunkl operators for Calogero–Moser–Sutherland and Ruijsenaars–Schneider type models.

**Remark.** In [187] (see also [188, 189]), the rather general classification for q-KZ flat connections was proposed. This classification is based on the interpretation of q-KZ flat connections (3.14.4) as images (in $R$-matrix representations) of commutative Jucys–Murphy elements for affine braid groups defined by Coxeter graphs. We discuss such braid groups below in Subsection **4.1**. In particular, the connections (3.14.7) are images of the Jucys–Murphy elements $\overline{J}_i$ for affine braid group $\mathcal{B}_N(C^{(1)})$ (see Proposition 2.1 in [187]).

### 3.15. Elliptic solutions of the Yang–Baxter equation

In this subsection, we consider $Z_N \otimes Z_N$-symmetric solutions of the Yang–Baxter equation (3.9.12) (see [190]). The elements $R^{i_1 i_2}_{j_1 j_2}(\theta)$ of the corresponding $R$-matrix will be expressed in terms of elliptic functions of the spectral parameter $\theta$.

We construct this solution explicitly, following the method of the paper [190]. We consider two matrices $g$ and $h$ such that $g^N = h^N = 1$:

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{3.15.1}$$

where $\omega = \exp(2\pi i/N)$ and $h\, g = \omega\, g\, h$. The matrices $g$ and $h$ are $Z_N$-graded generators of the algebra $\mathrm{Mat}(N)$, the graded basis for which can be chosen in the form

$$I_{\vec{\alpha}} = I_{\alpha_1 \alpha_2} = g^{\alpha_1} h^{\alpha_2}, \quad \alpha_{1,2} = 0, 1, \dots, N-1. \tag{3.15.2}$$

On the other hand, the matrices (3.15.2) realize a projective representation of the group $Z_N \otimes Z_N$: $I_{\vec{\alpha}} I_{\vec{\beta}} = \omega^{\alpha_2 \beta_1} I_{\vec{\alpha}+\vec{\beta}}$ . Any matrix $R_{12}(\theta) = R^{i_1 i_2}_{j_1 j_2}(\theta)$ can now be written in the form

$$R_{12}(\theta) = W_{\vec{\alpha},\vec{\beta}}(\theta)\, I_{\vec{\alpha}} \otimes I_{\vec{\beta}}$$

(the sum over $\alpha_i, \beta_j$ is assumed). We consider the $Z_N \otimes Z_N$-invariant subset of such matrices:

$$R_{12}(\theta) = W_{\vec{\alpha}}(\theta) I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}}, \tag{3.15.3}$$

where $I^{-1}_{\vec{\alpha}} = h^{-\alpha_2} g^{-\alpha_1} = \omega^{\alpha_1 \alpha_2} I_{-\vec{\alpha}}$. The invariance of the matrices (3.15.3) is expressed by the relations

$$R_{12}(\theta) = (I_{\vec{\gamma}} \otimes I_{\vec{\gamma}})\, R_{12}(\theta)\, (I_{\vec{\gamma}} \otimes I_{\vec{\gamma}})^{-1} \quad \forall \vec{\gamma}, \tag{3.15.4}$$

which obviously follow from the identity

$$I_{\vec{\gamma}}\, I_{\vec{\alpha}}\, I^{-1}_{\vec{\gamma}} = \omega^{<\alpha,\gamma>} I_{\vec{\alpha}}, \quad <\alpha,\, \gamma>= \alpha_1 \gamma_2 - \alpha_2 \gamma_1.$$

It was noted in [190] that the relations

$$\begin{gathered} R_{12}(\theta+1) = g_1^{-1}\, R_{12}(\theta)\, g_1 = g_2\, R_{12}(\theta)\, g_2^{-1}, \\ R_{12}(\theta+\tau) = \exp(-i\pi\tau) \exp(-2\pi i \theta) h_1^{-1}\, R_{12}(\theta)\, h_1 = \\ = \exp(-i\pi\tau) \exp(-2\pi i\theta) h_2\, R_{12}(\theta)\, h_2^{-1}, \end{gathered} \tag{3.15.5}$$

$$R_{12}(0) = I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}} \equiv P_{12}, \tag{3.15.6}$$

where $\tau$ is some complex parameter (period), are consistent with the Yang–Baxter equation (3.9.12) and can be regarded as subsidiary conditions to these equations (the last identity in (3.15.6) follows from $(I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}})\, I_{\vec{\beta}} \otimes I_{\vec{\gamma}} = I_{\vec{\gamma}} \otimes I_{\vec{\beta}}\, (I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}})$). Moreover, for the $Z_N \otimes Z_N$-invariant $R$-matrix (3.15.3) the conditions (3.15.5), (3.15.6) determine the solution of the Yang–Baxter equation uniquely. Indeed, substitution of (3.15.3) in (3.15.5), (3.15.6) leads to the equations

$$\begin{aligned} & W_{\vec{\alpha}}(\theta+1) = \omega^{\alpha_2}\, W_{\vec{\alpha}}(\theta), \\ & W_{\vec{\alpha}}(\theta+\tau) = \exp(-i\pi\tau)\, \exp(-2\pi i\theta)\, \omega^{-\alpha_1}\, W_{\vec{\alpha}}(\theta), \quad W_{\vec{\alpha}}(0) = 1, \end{aligned} \tag{3.15.7}$$

the solution of which can be found by means of an expansion in a Fourier series and has the form

$$W_{\vec{\alpha}}(\theta) = \frac{\Theta_{\vec{\alpha}}(\theta+\eta)}{\Theta_{\vec{\alpha}}(\eta)}, \quad (W_{\vec{\alpha}+\vec{\nu}}(u) = W_{\vec{\alpha}+\vec{\nu}'}(u) = W_{\vec{\alpha}}(u))\,, \tag{3.15.8}$$

where $\vec{\nu} = (N, 0)$, $\vec{\nu}' = (0, N)$,

$$\Theta_{\vec{\alpha}}(u) = \sum_{m=-\infty}^{\infty} \exp\left[ i\pi\tau(m + \frac{\alpha_2}{N})^2 + 2\pi i(m + \frac{\alpha_2}{N})(u + \frac{\alpha_1}{N})\right], \tag{3.15.9}$$

and we recall that $\alpha_{1,2} \in \mathbf{Z}_N$. The parameter $\eta$ in (3.15.8) is arbitrary. For $N = 2$, the solution (3.15.8) is identical to the solution obtained by Baxter [1, 3, 191] in connection with the investigation of the so-called eight-vertex lattice model.

Direct substitution of (3.15.3) in the Yang–Baxter equation (3.9.12) shows that the functions $W_{\vec{\alpha}}(\theta)$ must satisfy the relations

$$\sum_{\vec{\gamma}} W_{\vec{\gamma}}(\theta-\theta')\, W_{\vec{\alpha}-\vec{\gamma}}(\theta)\, W_{\vec{\beta}+\vec{\gamma}}(\theta') \left(\omega^{<\gamma,\beta>} - \omega^{<\alpha-\gamma,\beta>}\right) = 0. \tag{3.15.10}$$

As it was proved in [192–195], these relations hold when the functions (3.15.8) and (3.15.9) are substituted. We will see later that the identities (3.15.10) are intimately related to a version of the Yang–Baxter equations appeared in Interaction Round Face models (see Subsection **5.3**).

**Remark.** In the paper [196], $(Z_N \times Z_N)$-invariant solutions $R(\theta)$ of the Yang–Baxter equation are interpreted as matrix analogues of elliptic functions. A matrix analogue of the Weierstrass sigma function $\sigma(\theta)$ is introduced in [196], which is an entire matrix function with zeros at the points of a 2D lattice $L$ and satisfies quasiperiodicity conditions similar to (3.15.5). Then the $(Z_N \times Z_N)$-invariant $R$-matrix is constructed as the ratio

$$R(\theta) = \sigma^{-1}(\theta+\eta)\,\sigma(\theta-\eta), \tag{3.15.11}$$

where $\eta$ is an additional parameter. It turns out (see [196]) that the representation (3.15.11) remains valid with degeneracy of the lattice $L$ to a one-dimensional lattice (the trigonometric case) or to a zero-dimensional lattice (the rational case). In the latter case, the sigma function is chosen in the form $\sigma_0(\theta) = P_+ + \theta P_-$ (here $P_\pm = \frac{1}{2}(I \pm P)$), i.e., it is represented as the polynomial of the first order in $\theta$. Interestingly, there is an inverse procedure when the complete elliptic Weierstrass matrix function $\sigma(\theta)$ and also the $R$-matrix (3.15.11) can be obtained by using a special multiplicative averaging of matrices $\sigma_0(\theta)$ and $R_0 = \sigma_0^{-1}(\theta+\eta)\,\sigma_0(\theta-\eta)$ over the lattice $L$. Finally, we note that for trigonometric solutions $R(\theta)$ analogous representations, as a product of ratios of entire matrix functions, were studied in detail in [197]. The entire matrix functions introduced in [197] can be considered as matrix generalizations of the trigonometric functions.

In connection with this remark, we also recall the representations (3.8.6), (3.12.15) in the form of rational functions for trigonometric solutions $R$ (in defining representations of $SL_q(N), SO_q(N)$, and $Sp_q(2n)$), obtained by using the Baxterization procedure. For Yangian solutions the analogous formula is given in Eq. (3.13.56).

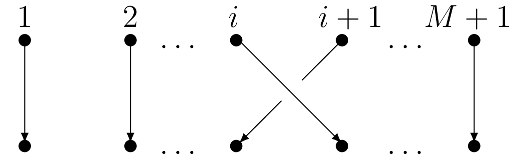
## 4. Group algebra of braid group and its quotients

### *4.1. Affine braid groups and Coxeter graphs*

A braid group $\mathcal{B}_{M+1}$ is generated by elements $\sigma_i$ $(i = 1, \dots M)$ subject to the relations

$$\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}, \quad [\sigma_i, \sigma_j] = 0 \;\text{ for }\; |i-j| > 1. \tag{4.1.1}$$

By definition the elements $\sigma_i$ are supposed to be invertible and represented graphically as (cf. (3.1.14)):

$\sigma_i \;=$ 


(4.1.2)

**Definition 13.** *An extension of the braid group $\mathcal{B}_M$ by one invertible generator $\sigma_M$ subject to the relations*

$$\sigma_M\sigma_{M-1}\sigma_M = \sigma_{M-1}\sigma_M\sigma_{M-1}, \quad \sigma_1\sigma_M\sigma_1 = \sigma_M\sigma_1\sigma_M, \\ [\sigma_M,\, \sigma_k] = 0 \;\; (k = 2, \dots, M-2) \tag{4.1.3}$$

*is called a periodic braid group $\overline{\mathcal{B}}_M \equiv \mathcal{B}_M(A^{(1)})$.*

**Definition 14.** *An extension of the braid group $\mathcal{B}_{M+1}$ $(M \geqslant 1)$ by one invertible generator $y_1$ which satisfies the relations*

$$y_1\sigma_1 y_1\sigma_1 = \sigma_1 y_1\sigma_1 y_1, \quad [\sigma_i,\, y_1] = 0 \;\; \forall i > 1 \tag{4.1.4}$$

*is called the affine braid group $\hat{\mathcal{B}}_{M+1} \equiv \mathcal{B}_{M+1}(C)$ of the $C$ type. The extension of the group $\hat{\mathcal{B}}_{M+1}$ by one more additional generator $y_{M+1}$ with the defining relations*

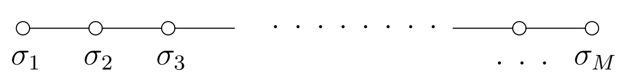
$$y_{M+1}\sigma_M y_{M+1}\sigma_M = \sigma_M y_{M+1}\sigma_M y_{M+1}, \quad [\sigma_i,\, y_{M+1}] = 0 \;\; \forall i < M, \quad [y_1,\, y_{M+1}] = 0 \tag{4.1.5}$$

*is the affine braid group of the $C^{(1)}$ type which is denoted as $\mathcal{B}_{M+1}(C^{(1)})$.*

It is clear that the affine group $\mathcal{B}_{M+1}(C)$ is the subgroup of the affine braid group $\mathcal{B}_{M+1}(C^{(1)})$, while the braid group $\mathcal{B}_{M+1}$ is the subgroup of $\mathcal{B}_{M+1}(C)$. The defining relations (4.1.1), (4.1.3), (4.1.4), and (4.1.5) (where we denote $y_1 = \sigma_0$ and $y_{M+1} = \sigma_{M+1}$) of the (affine) braid groups can be written in the unified form as

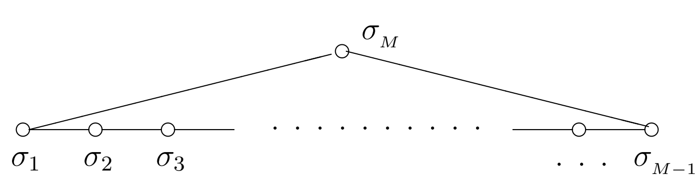
$$\underbrace{\sigma_i\,\sigma_j\,\sigma_i\;\dots}_{m_{ij}\ \text{factors}} = \underbrace{\sigma_j\,\sigma_i\,\sigma_j\;\dots}_{m_{ij}\ \text{factors}}, \tag{4.1.6}$$

where $m_{ij} = m_{ji}$ are integers such that $m_{ii} = 1$, $m_{ij} \geqslant 2$ for $i \neq j$. The set of data given by the matrix $||m_{ij}||$ is conveniently represented as the *Coxeter graph* with $M$ (or $M+1$, or $M+2$) nodes associated with generators $\sigma_i$, and the nodes $i$ and $j$ are connected by $(m_{ij} - 2)$ lines if $m_{ij} = 2, 3, 4$ and by 3 lines if $m_{ij} = 6$. Thus, the Coxeter graph for the braid group relations (4.1.1) is the $A$-type graph:




(4.1.7)

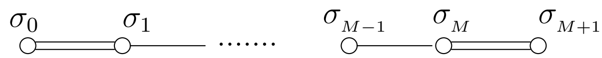
and, for the affine braid group relations (4.1.3), (4.1.4), and (4.1.5), the Coxeter graphs are respectively

$A^{(1)} \;=$ 


(4.1.8)

$C$ = 

(4.1.9)

$C^{(1)}$ = 

(4.1.10)

In the same way, one can also define the affine braid groups of $B^{(1)}$ and $D^{(1)}$ types (see, e.g., [187] and references therein).

Consider the affine braid group $\hat{\mathcal{B}}_{M+1} \equiv \mathcal{B}_{M+1}(C)$. The elements $\{y_i\}$ $(i = 1, \dots M+1)$ defined by

$$y_1, \quad y_2 = \sigma_1 y_1 \sigma_1, \quad y_3 = \sigma_2 \sigma_1 y_1 \sigma_1 \, \sigma_2, \; \dots, \; y_{i+1} = \sigma_i y_i \sigma_i, \tag{4.1.11}$$

are called *Jucys–Murphy elements* and generate an Abelian subgroup in $\hat{\mathcal{B}}_{M+1}$. For $y_1 = 1$, the Jucys–Murphy elements (4.1.11) generate an Abelian subgroup in the braid group $\mathcal{B}_{M+1}$. Note that condition $y_n y_{n+1} = y_{n+1} y_n$ is equivalent to the reflection equation for $y_n$:

$$y_n \, \sigma_n \, y_n \, \sigma_n = \sigma_n \, y_n \, \sigma_n \, y_n. \tag{4.1.12}$$

Then we have $y_n \, y_{n+1} \, \sigma_n = \sigma_n \, y_n \, y_{n+1}$, and the element $Z = y_1 y_2 \cdots y_{M+1}$ is obviously central in $\mathcal{B}_{M+1}$.

**Proposition 4.16.** *The product of $m$ elements of $\hat{B}_{M+1}$: $y^{(m)}_{k+1} := y_{k+1} y_{k+2} \dots y_{k+m}$ $(k+m < M+1)$ satisfies the following relations:*

$$y^{(m)}_{k+1} = U_{(k,m)} \, y^{(m)}_1 \, U_{(m,k)}, \tag{4.1.13}$$

*where (cf. (3.2.46))*

$$U_{(k,m)} = \sigma_{(k \to m+k-1)} \dots \sigma_{(2 \to m+1)} \, \sigma_{(1 \to m)} \equiv \sigma_{(k \leftarrow 1)} \sigma_{(k+1 \leftarrow 2)} \sigma_{(k+m-1 \leftarrow m)}, \tag{4.1.14}$$

*and* $(k \leqslant n)$

$$\sigma_{(k \to n)} = \sigma_k \sigma_{k+1} \dots \sigma_n, \quad \sigma_{(n \leftarrow k)} = \sigma_n \dots \sigma_{k+1} \sigma_k.$$

**Proof.** First of all, we show that

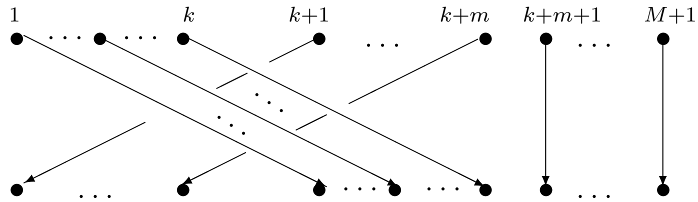
$$y^{(m)}_{k+1} = \sigma_{(k \to k+m-1)} \, y^{(m)}_k \, \sigma_{(k+m-1 \leftarrow k)}. \tag{4.1.15}$$

This identity is proved by induction. For $m = 1$ we obviously have $y_{k+1} = \sigma_k y_k \sigma_k$. Let (4.1.15) be correct for some $m$. Then, for $y^{(m+1)}_{k+1}$ we have

$$\begin{aligned} y^{(m+1)}_{k+1} &= y^{(m)}_{k+1} \, y_{k+m+1} = \sigma_{(k \to k+m-1)} \, y^{(m)}_k \, \sigma_{(k+m-1 \leftarrow k)} \, \sigma_{k+m} y_{k+m} \sigma_{k+m} = \\ &= \sigma_{(k \to k+m-1)} \, y^{(m)}_k \, \sigma_{k+m} y_{k+m} \sigma_{k+m} \, \sigma_{(k+m-1 \leftarrow k)} = \\ &= \sigma_{(k \to k+m-1)} \sigma_{k+m} \, (y^{(m)}_k y_{k+m}) \, \sigma_{k+m} \sigma_{(k+m-1 \leftarrow k)}, \end{aligned}$$

which coincides with (4.1.15) for $m \to m+1$. Applying (4.1.15) several times, we deduce (4.1.13). ■

One can graphically represent elements $U_{(k,m)}$ (4.1.14) (by means of the rules (4.1.2)) in the following form (cf. (3.1.61)):

$U_{(k,m)} \;=$ 


(4.1.16)

From this representation it becomes clear that the following braid relations hold:

$$U_{(k,m)}\,(T_m U_{(k,n)})\,U_{(m,n)} = (T_k U_{(m,n)})\,U_{(k,n)}\,(T_n U_{(k,m)}), \tag{4.1.17}$$

where we have introduced jump operations $T_m$: $\sigma_i \to \sigma_{i+m}$. One can check relations (4.1.17) by direct calculations.

**Remark.** The sets of commutative Jucys–Murphy elements for all affine braid groups of the $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ and $D^{(1)}$ types were constructed in [187]. These elements were realized in $R$-matrix representations of Birman–Murakami–Wenzl algebras and then used in [187] for the formulation of the special q-KZ equations.

*4.2. Group algebra of the braid group $B_{M+1}$ and shuffle elements*

We denote the group algebra of the braid group $B_{M+1}$ over complex numbers as $\mathbb{C}[B_{M+1}]$. Consider the elements $\Sigma_{m\to n} \in \mathbb{C}[B_{M+1}]$, for $n = m, m+1, \ldots, M+1$, that are defined inductively

$$\Sigma_{m\to n} = f_{m\to n}\,\Sigma_{m\to n-1} = f_{m\to n}\,f_{m\to n-1}\cdots f_{m\to m+1}\,f_{m\to m}, \tag{4.2.1}$$

where the subscript $m \to n$ denotes the set of indices $(m, m+1, \ldots, n)$, $\Sigma_{m\to m} = 1$ and

$$\begin{aligned} f_{k\to k} = 1, \quad f_{k\to n} &= 1 + \sigma_{n-1} + \sigma_{n-2}\,\sigma_{n-1} + \cdots + \sigma_k\,\sigma_{k+1}\cdots\sigma_{n-1} \;= \\ &= 1 + f_{k\to n-1}\,\sigma_{n-1}, \qquad k < n. \end{aligned} \tag{4.2.2}$$

Note that, by means of braid relations (4.1.1), we derive the mirror set of expressions for the elements $\Sigma_{k\to n}$ (4.2.1):

$$\Sigma_{k\to n} = \Sigma_{k\to n-1}\,\overline{f}_{k\to n} = \overline{f}_{k\to k}\,\overline{f}_{k\to k+1}\cdots\overline{f}_{k\to n-1}\,\overline{f}_{k\to n}, \tag{4.2.3}$$

where

$$\begin{aligned} \overline{f}_{k\to k} = 1, \quad \overline{f}_{k\to n} &= 1 + \sigma_{n-1} + \sigma_{n-1}\,\sigma_{n-2} + \cdots + \sigma_{n-1}\ldots\sigma_{k+1}\,\sigma_k = \\ &= 1 + \sigma_{n-1}\overline{f}_{k\to n-1}\,, \qquad k < n. \end{aligned} \tag{4.2.4}$$

The elements $\Sigma_{m+1\to n}$ play the role of symmetrizers in $\mathbb{C}[B_{M+1}]$ and, in view of the projection $\sigma_i \to 1$ for (4.2.1)–(4.2.4), they are algebraic analogs of the factorials $(n-m)!$. The important properties of the elements $f_{k\to n} \in \mathbb{C}[B_{M+1}]$ are

$$f_{1\to n}f_{1\to n-1}\ldots f_{1\to m+1} = \text{Ш}^{(m,n-m)}_{1\to n}\,\Sigma_{m+1\to n} \quad (0 \leqslant m < n), \tag{4.2.5}$$

where we introduce elements $\text{Ш}^{(m,n-m)}_{1\to n} \in \mathbb{C}[B_{M+1}]$ with initial conditions $\text{Ш}^{(n,0)}_{1\to n} = \text{Ш}^{(0,n)}_{1\to n} = 1$, $\text{Ш}^{(m-1,1)}_{1\to m} = f_{1\to m}$. The identities (4.2.5) and definition (4.2.1), written in the form

$$\Sigma_{1\to n} = f_{1\to n}f_{1\to n-1}\ldots f_{1\to m+1}\Sigma_{1\to m},$$

lead to the right factorization formula

$$\Sigma_{1\to n} = Ш^{(m,n-m)}_{1\to n}\, \Sigma_{1\to m}\, \Sigma_{m+1\to n}. \tag{4.2.6}$$

Thus, taking into account the interpretation of $\Sigma_{m\to n}$ as factorials, one can consider $Ш^{(m,n-m)}_{1\to n}$ as an algebraic analog of the binomial coefficient.

**Proposition 4.17.** *The elements $Ш^{(m,n-m)}_{1\to n}$ are defined inductively by using the recurrent relations [47, 199] (braid analogs of the Pascal rule):*

$$Ш^{(m,n+1-m)}_{1\to n+1} = Ш^{(m,n-m)}_{1\to n} + Ш^{(m-1,n+1-m)}_{1\to n}\, \sigma_n\, \sigma_{n-1} \dots \sigma_m. \tag{4.2.7}$$

**Proof.** We denote $f_k := f_{1\to k}$ and consider formula (4.2.5) for $n \to (n+1)$:

$$\begin{gathered} Ш^{(m,n+1-m)}_{1\to n+1}\, \Sigma_{m+1\to n+1} = f_{n+1} f_n \cdots f_{m+1} = (1 + f_n \sigma_n) f_n f_{n-1} \cdots f_{m+1} = \\ = f_n f_{n-1} \cdots f_{m+1} + f_n \sigma_n (1 + f_{n-1}\sigma_{n-1}) f_{n-1} f_{n-2} \cdots f_{m+1} = \\ = f_n \cdots f_{m+1}(1+\sigma_n) \;+\; f_n f_{n-1} \sigma_n \sigma_{n-1} (1 + f_{n-2}\sigma_{n-2}) f_{n-2} \cdots f_{m+1} = \cdots = \\ = f_n \cdots f_{m+1}(1 + \sigma_n + \sigma_n\sigma_{n-1} + \cdots + \sigma_n \cdots \sigma_{m+1}) + f_n \cdots f_m\, \sigma_n \cdots \sigma_m = \\ = Ш^{(m,n-m)}_{1\to n}\, \Sigma_{m+1\to n}\, \overline{f}_{m+1\to n+1} + Ш^{(m-1,n-m+1)}_{1\to n}\, \Sigma_{m\to n}\, \sigma_n \sigma_{n-1} \cdots \sigma_m = \\ = Ш^{(m,n-m)}_{1\to n}\, \Sigma_{m+1\to n+1} \;+ Ш^{(m-1,n-m+1)}_{1\to n}\, \sigma_n \sigma_{n-1} \cdots \sigma_m\, \Sigma_{m+1\to n+1}, \end{gathered} \tag{4.2.8}$$

where we used (4.2.2)–(4.2.4). After dividing both sides of (4.2.8) by $\Sigma_{m+1\to n+1}$ from the right, we obtain (4.2.7). ■

In particular, Eq. (4.2.7) is written for $m = (n-1)$ as $Ш^{(n-1,2)}_{1\to n+1} = (f_{1\to n} + Ш^{(n-2,2)}_{1\to n}\, \sigma_n\, \sigma_{n-1})$, $(n \geqslant 2)$ which gives

$$\begin{gathered} Ш^{(m-1,2)}_{1\to m+1} = f_{1\to m} + f_{1\to m-1}\, (\sigma_m\, \sigma_{m-1}) + f_{1\to m-2}\, (\sigma_{m-1}\, \sigma_{m-2})\, (\sigma_m\, \sigma_{m-1}) + \\ + \cdots + f_{1\to 2}\, (\sigma_3\, \sigma_2) \cdots (\sigma_m\, \sigma_{m-1}) + (\sigma_2\, \sigma_1) \cdots (\sigma_m\, \sigma_{m-1}). \end{gathered} \tag{4.2.9}$$

The next relation is $Ш^{(n-2,3)}_{1\to n+1} = Ш^{(n-2,2)}_{1\to n} + Ш^{(n-3,3)}_{1\to n}\, \sigma_n\, \sigma_{n-1}\, \sigma_{n-2}$ for $(n \geqslant 3)$, etc.

Note that $Ш^{(m,n-m)}_{1\to n}$ are sums over the braid group elements which can be considered as quantum analogs of $(m,\, n-m)$ shuffles of two piles with $m$ and $(n-m)$ cards if we read all monomials in $Ш^{(m,n-m)}_{1\to n}$ from right to left (the standard shuffles are obtained by projection $\sigma_i \to s_i$, where $s_i$ are generators of the symmetric group $\mathcal{S}_{M+1}$). As it follows from (4.2.5), the elements $f_{1\to m} = Ш^{(m-1,1)}_{1\to m}$ are the sums of $(m-1, 1)$ shuffles. One can use the operators $\Sigma_{1\to n}$ (4.2.1) and identities (4.2.6) for the definition of the associative products that are analogs of the wedge products proposed by S. Woronowicz in the theory of differential calculus on quantum groups [70]. In view of (4.2.6), these products are related to the quantum shuffle products (for quantum shuffles and corresponding products, see [198, 199]). The associativity of these products is provided by the identities

$$Ш^{(n-m,m)}_{1\to n}\, Ш^{(k,m-k)}_{1\to m} = Ш^{(k,n-k)}_{1\to n}\, Ш^{(m-k,n-m)}_{k+1\to n} \quad (k < m < n), \tag{4.2.10}$$

which are the consistence conditions for the definition of a 3-pile shuffles $(k, m-k, n-m)$:

$$\Sigma_{1\to n} = Ш^{(k,m-k,n-m)}_{1\to n}\, \Sigma_{1\to k}\, \Sigma_{k+1\to m}\, \Sigma_{m+1\to n}.$$

Going further, one can introduce $m$-pile shuffles $Ш_{1\to n}^{(n_1,n_2,\dots,n_m)}$ of the pack of $n$ cards ($n = n_1+n_2+\cdots+n_m$). Then we observe that the "symmetrizer" $\Sigma_{1\to n}$ (4.2.1) is nothing else but the $n$-pile shuffle $Ш_{1\to n}^{(1,1,\dots,1)}$.

By means of the mirror mapping (when we write generators $\sigma_k \in \mathcal{B}_{M+1}$ in all monomials in opposite order) we obtain from (4.2.6) a left factorization formula

$$\Sigma_{1\to n} = \Sigma_{1\to m}\,\Sigma_{m+1\to n}\,\overline{Ш}_{1\to n}^{(m,n-m)}, \tag{4.2.11}$$

where the elements $\overline{Ш}_{1\to n}^{(m,n-m)}$ are defined by recurrence relations [47, 199] (cf. (4.2.7))

$$\overline{Ш}_{1\to n+1}^{(m,n-m+1)} = \overline{Ш}_{1\to n}^{(m,n-m)} + \sigma_m\dots\sigma_n\,\overline{Ш}_{1\to n}^{(m-1,n-m+1)}$$

with initial conditions $\overline{Ш}_{1\to n}^{(n,0)} = \overline{Ш}_{1\to n}^{(0,n)} = 1$, $\overline{Ш}_{1\to n}^{(n-1,1)} = \overline{f}_{1\to n}$, and $\overline{Ш}_{1\to n}^{(m,n-m)}$ is a sum over $(m, n-m)$ quantum shuffles (if we read all monomials from left to right). The mirror analogs of the factorization identities (4.2.10) also hold

$$\overline{Ш}_{1\to m}^{(k,m-k)}\,\overline{Ш}_{1\to n}^{(m,n-m)} = \overline{Ш}_{k+1\to n}^{(m-k,n-m)}\,\overline{Ш}_{1\to n}^{(k,n-k)}.$$

### *4.3. A-type Hecke algebra $H_{M+1}(q)$*

#### *4.3.1. Jucys–Murphy elements, symmetrizers and antisymmetrizers in $H_{M+1}$*

$A$-type Hecke algebra $H_{M+1}(q)$ (see, e.g., [201] and references therein) is a quotient of the braid group algebra (4.1.1) by the additional relation

$$\sigma_i^2 - 1 = \lambda\,\sigma_i, \quad (i = 1,\dots,M). \tag{4.3.1}$$

Here $\lambda = (q-q^{-1})$, and $q \in \mathbb{C}\backslash\{0,\pm1\}$ is a deformation parameter. Note that algebras $H_{M+1}(q)$ and $H_{M+1}(-q^{-1})$ are isomorphic to each other: $H_{M+1}(q) \simeq H_{M+1}(-q^{-1})$. The group algebra of $\mathcal{B}_{M+1}$ (4.1.1) has an infinite dimension, while its quotient $H_{M+1}(q)$ is finite-dimensional. It can be shown (see, e.g., [203]) that $H_{M+1}(q)$ is linearly spanned by $(M+1)!$ monomials appeared in the expansion of $\Sigma_{1\to M+1}$ (4.2.1) (or in the expansion of (4.2.3)).

The $A$-type Hecke algebra is a special case of a general affine Hecke algebra. The affine Hecke algebra is the quotient (by additional constraint (4.3.1)) of the affine braid groups with generators $\{\sigma_i\}$ subject to general relations (4.1.6). As it was shown in Subsection **4.1**, the Coxeter graph for the braid group relations (4.1.1) is the $A$-type graph (4.1.7). That is why the Hecke algebra with defining relations (4.1.1) and (4.3.1) is called the $A$-type Hecke algebra. The $A$-type Hecke algebra $H_{M+1}(q)$ is a semisimple algebra.

An essential information about a finite-dimensional semisimple associative algebras $\mathcal{A}$ is contained (see, e.g., [138]; see also Subsection 4.5 in [139] and references therein) in the structure of its regular bimodule, which is decomposed into direct sums:

$$\mathcal{A} = \bigoplus_{\alpha=1}^{s}\mathcal{A}\cdot e_\alpha, \quad \mathcal{A} = \bigoplus_{\alpha=1}^{s} e_\alpha\cdot\mathcal{A}$$

of left and right submodules (ideals), respectively (*left and right Peirce decompositions*). Here the elements $e_\alpha \in \mathcal{A}$ ($\alpha = 1,\dots,s$) are mutually orthogonal idempotents resolving the identity operator 1:

$$e_\alpha\cdot e_\beta = \delta_{\alpha\beta}e_\alpha, \qquad 1 = \sum_{\alpha=1}^{s} e_\alpha. \tag{4.3.2}$$

Making use of left and right Peirce decompositions simultaneously, we have *two-sided Peirce decomposition*

$$\mathcal{A} = \bigoplus_{\alpha,\beta=1}^{s} e_\alpha\,\mathcal{A}\,e_\beta = \bigoplus_{\alpha,\beta=1}^{s} \mathcal{A}_{\alpha,\beta}, \qquad \mathcal{A}_{\alpha,\beta} = e_\alpha\,\mathcal{A}\,e_\beta. \tag{4.3.3}$$

Here the linear spaces $\mathcal{A}_{\alpha,\beta}$ are, generally speaking, neither left nor right ideals in $\mathcal{A}$. Instead, products of elements of $\mathcal{A}_{\alpha,\beta}$ obey the relations: $\mathcal{A}_{\alpha,\beta}\cdot\mathcal{A}_{\gamma,\kappa} = \delta_{\beta,\gamma}\,\mathcal{A}_{\alpha,\kappa}$, which resemble relations for matrix units.

The number $s$ depends on the choice of the type of idempotents in $\mathcal{A}$. There are two important types of the idempotents in $\mathcal{A}$ and correspondingly two decompositions of the identity operator:
(1) *Primitive idempotents.* An idempotent $e_\alpha$ is primitive if it cannot be further resolved into a sum of nontrivial mutually orthogonal idempotents.
(2) *Primitive central idempotents.* An idempotent $e'_A$ $(A = 1,\dots,s')$ is primitive central if it is central element in $\mathcal{A}$ and primitive in the class of central idempotents.
One can expand any central idempotent $e_A$ in primitive idempotents $\{e_\alpha\}$: $e_A = \sum_{\alpha\in A} e_\alpha$, where $A$ is a subset of indices from the set $\{1,2,\dots,s\}$; i.e., central orthogonal idempotents $\{e_A\}$ are conveniently labeled by non-intersecting subsets $A \subset \{1,2,\dots,s\}$, which cover the entire set of indices $\{1,2,\dots,s\}$.

Let $A_i$ $(i = 1,\dots,s')$ be non-intersecting subsets in $\{1,2,\dots,s\}$ which cover the entire set and define central idempotents $e_{A_i}$. Let $\alpha \in A_i$, $\beta \in A_j$ and $i \neq j$, then, in view of orthogonality $e_{A_i}\cdot e_{A_j} = 0$, for any element $a \in \mathcal{A}$ we have $a_{\alpha,\beta} = e_\alpha\cdot a\cdot e_\beta = 0$. This tells us that the two-sided Peirce decomposition (4.3.3) of semisimple algebra $\mathcal{A}$ does not contain terms $\mathcal{A}_{\alpha,\beta}$, if $\alpha \in A_i$, $\beta \in A_j$ and $i \neq j$, so that we have

$$\mathcal{A} = \bigoplus_{i=1}^{s'} e'_{A_i}\cdot\mathcal{A}\cdot e'_{A_i} = \bigoplus_{i=1}^{s'} \bigoplus_{\alpha,\beta\in A_i} e_\alpha\,\mathcal{A}\,e_\beta = \bigoplus_{i=1}^{s'} \bigoplus_{\alpha,\beta\in A_i} \mathcal{A}_{\alpha,\beta}, \tag{4.3.4}$$

where, again, $s'$ is the number of primitive central idempotents $e_{A_i}$. Thus, the regular bimodule of the semisimple algebra $\mathcal{A}$ decomposes into direct sums of irreducible sub-bimodules (two-sided ideals) $\mathcal{A} = \bigoplus_{i=1}^{s'} \mathcal{A}\cdot e'_{A_i} = \bigoplus_{i=1}^{s'} e'_{A_i}\cdot\mathcal{A}$ with respect to the central idempotents $e'_A$. For semisimple algebras $\mathcal{A}$ the subspaces $\mathcal{A}_{\alpha,\beta}$ in (4.3.4) are one-dimensional and for any $a \in \mathcal{A}$ we have $e_\alpha\cdot a\cdot e_\beta = c(a)\,e_{\alpha\beta}$, where $c(a)$ are constants and basis elements $e_{\alpha\beta} \in \mathcal{A}_{\alpha,\beta}$ are normalized such that $e_{\alpha\beta}\cdot e_{\gamma\delta} = \delta_{\beta\gamma}\,e_{\alpha\delta}$. In view of these relations, the elements $e_{\alpha\beta} \in \mathcal{A}$ are called matrix units. The diagonal matrix units coincide with the primitive idempotents: $e_{\alpha\alpha} = e_\alpha$.

Now we return back to the consideration of the Hecke algebra $H_{M+1}$ (here and below we omit the parameter $q$ in the notation $H_{M+1}(q)$). First of all, we construct two special primitive idempotents in the Hecke algebra $H_{M+1}$ which correspond to the symmetrizers and antisymmetrizers. For this purpose, we consider two substitutions $\sigma_i \to q\sigma_i$, $\sigma_i \to -q^{-1}\sigma_i$ for the braid group algebra element $\Sigma_{1\to n}$ (4.2.1). As a result, for the algebra $H_{M+1}$ we obtain two sequences of operators $S_{1\to n}$ and $A_{1\to n}$ $(n = 1,\dots M+1)$:

$$S_{1\to n} := a_n^-\,\Sigma_{1\to n}(q\,\sigma_i), \quad A_{1\to n} := a_n^+\,\Sigma_{1\to n}(-q^{-1}\,\sigma_i) \tag{4.3.5}$$

$$\left( a_n^{\mp} = \frac{q^{\mp\frac{n(n-1)}{2}}}{[n]_q!}, \quad [n]_q! := [1]_q\,[2]_q\cdots[n]_q, \quad [n]_q = \frac{(q^n - q^{-n})}{(q - q^{-1})} \right),$$

$$\sigma_i\, S_{1\to n} = S_{1\to n}\, \sigma_i = q\, S_{1\to n} \;\; (i = 1, \dots n-1), \tag{4.3.6}$$
$$\sigma_i\, A_{1\to n} = A_{1\to n}\, \sigma_i = -\tfrac{1}{q}\, A_{1\to n} \;\; (i = 1, \dots n-1),$$

which are symmetrizers and antisymmetrizers, respectively (see [111]). The normalization factors $a_n^{\mp}$ have been introduced in (4.3.5) in order to obtain the idempotent conditions $S_{1\to n}^2 = S_{1\to n}$ and $A_{1\to n}^2 = A_{1\to n}$. Here we additionally suppose that $[n]_q \neq 0$, $\forall n = 1, \dots, M+1$. The first two idempotents are

$$S_{12} = \frac{1}{[2]_q}(q^{-1} + \sigma_1), \quad A_{12} = \frac{1}{[2]_q}(q - \sigma_1). \tag{4.3.7}$$

Note that Eqs. (4.3.6) immediately follow from the factorization relations (4.2.6), (4.2.11), the form of the first idempotents (4.3.7) and the Hecke condition (4.3.1). The projectors $S_{1\to n}$ and $A_{1\to n}$ (4.3.5) correspond to the Young tableaux which have only one row and one column

$$P\left(\boxed{1}\boxed{\;\dots\;}\boxed{n}\right) = S_{1\to n}, \quad P\left(\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline \end{array}\right) = A_{1\to n}.$$

It follows directly from (4.3.6) that the idempotents $S_{1\to M+1}$ and $A_{1\to M+1}$ are central in $H_{M+1}(q)$.

Consider now the elements $y_i$ $(i = 1, \dots, M+1)$ (4.1.11) which generate a commutative subalgebra $Y_{M+1}$ in $H_{M+1}$. It can be proved that $Y_{M+1}$ is a maximal commutative subalgebra in $H_{M+1}$. The elements $y_i$ are called *Jucys–Murphy elements* and can be easily rewritten in the form (by using the Hecke condition (4.3.1) and braid relations (4.1.1)):

$$\begin{aligned} y_1 = 1, \quad y_i \;&=\; \sigma_{i-1}\, y_{i-1}\, \sigma_{i-1} \;=\; \sigma_{i-1} \dots \sigma_2\, \sigma_1^2\, \sigma_2 \dots \sigma_{i-1} = \\ &= \lambda\sigma_{i-1} \dots \sigma_2\, \sigma_1\, \sigma_2 \dots \sigma_{i-1} + \sigma_{i-1} \dots \sigma_3\, \sigma_2^2\, \sigma_3 \dots \sigma_{i-1} = \dots = \\ &= \lambda \sum_{k=1}^{i-2} \sigma_{i-1} \dots \sigma_{k+1}\, \sigma_k\, \sigma_{k+1} \dots \sigma_{i-1} + \lambda\sigma_{i-1} + 1 = \\ &= \lambda \sum_{k=1}^{i-2} \sigma_k \dots \sigma_{i-2}\, \sigma_{i-1}\, \sigma_{i-2} \dots \sigma_k + \lambda\sigma_{i-1} + 1, \qquad i = 2, \dots, M+1. \end{aligned} \tag{4.3.8}$$

It is interesting that the idempotents (4.3.5) which correspond to the symmetrizers and antisymmetrizers (the Young tableaux are only one row or column) can be constructed in the different way as polynomial functions of the elements $y_n$.

**Proposition 4.18.** *The idempotents $S_{1\to n}$ and $A_{1\to n}$ $(n = 2, \dots M+1)$ (4.3.5) are expressed in terms of the Jucys–Murphy elements as*

$$S_{1\to n} = \frac{(y_2 - q^{-2})}{(q^2 - q^{-2})}\, \frac{(y_3 - q^{-2})}{(q^4 - q^{-2})} \cdots \frac{(y_n - q^{-2})}{(q^{2(n-1)} - q^{-2})}, \tag{4.3.9}$$

$$A_{1\to n} = \frac{(y_2 - q^{2})}{(q^{-2} - q^{2})}\, \frac{(y_3 - q^{2})}{(q^{-4} - q^{2})} \cdots \frac{(y_n - q^{2})}{(q^{2(1-n)} - q^{2})}. \tag{4.3.10}$$

**Proof.** We note that expressions (4.3.7) for the first two projectors are written as

$$S_{12} = \frac{(\sigma_1^2 - q^{-2})}{(q^2 - q^{-2})}, \quad A_{12} = \frac{(\sigma_1^2 - q^{2})}{(q^{-2} - q^{2})},$$

and therefore Eqs. (4.3.9) and (4.3.10) are valid for $n = 2$. We prove Eqs. (4.3.9) and (4.3.10) by induction. Let Eqs. (4.3.9) and (4.3.10) be correct for some $n = k$. We need to prove these equations for $n = k + 1$ or we have to show that

$$S_{1\to k+1} = S_{1\to k} \cdot \frac{(y_{k+1} - q^{-2})}{(q^{2k} - q^{-2})}, \quad A_{1\to k+1} = A_{1\to k} \cdot \frac{(y_{k+1} - q^{2})}{(q^{-2k} - q^{2})}. \tag{4.3.11}$$

We prove only the first equation in (4.3.11) (the proof of the second equation in (4.3.11) is analogous). We substitute in (4.3.11) the last expression for Jucys–Murphy elements $y_{k+1}$ (4.3.8) and take into account (4.3.6). As a result, we obtain for the first equation in (4.3.11):

$$\begin{aligned} S_{1\to k+1} &= \tfrac{1}{(q^{2k}-q^{-2})}\, S_{1\to k} \left(\lambda(q^{k-1}\sigma_k \dots \sigma_1 + q^{k-2}\sigma_k \dots \sigma_2 + \dots + \sigma_k) + 1 - q^{-2}\right) = \\ &= \tfrac{q^k}{[k+1]_q}\, S_{1\to k} \left(q^{k}\sigma_k \dots \sigma_1 + q^{k-1}\sigma_k \dots \sigma_2 + \dots + q\,\sigma_k + 1\right), \end{aligned} \tag{4.3.12}$$

which coincides with the definition of symmetrizers (4.2.3), (4.3.5). This ends the proof of the induction and hence this Proposition. ■

Note that the idempotents $S_{1\to M+1}$ and $A_{1\to M+1}$ are central in the algebra $H_{M+1}(q)$ and represented as the polynomials $\sim (y_2 - t)(y_3 - t)\cdots(y_{M+1} - t)$, where $t = q^{\mp 2}$ (see (4.3.9), (4.3.10)), which are symmetric functions in variables $\{y_i\}$ $(i = 2, \dots, M + 1)$. In view of this, one can conjecture that all symmetric functions in $y_i$ generate the central subalgebra $Z_{M+1}$ in the Hecke algebra $H_{M+1}(q)$. Indeed, to prove this fact, we need only to check the relations: $[\sigma_k,\, y_n + y_{n+1}] = 0 = [\sigma_k,\, y_n y_{n+1}]$ for all $k < n + 1$.

New identities for the elements $y_i$ follow from the representations (4.3.9) and (4.3.10) (if we use Eqs. (4.3.6)):

$$(y_i - q^{2(i-1)})\, S_{1\to n} = 0 \Rightarrow (y_i - q^{2(i-1)})(y_2 - q^{-2})(y_3 - q^{-2}) \cdots (y_n - q^{-2}) = 0, \tag{4.3.13}$$

$$(y_i - q^{2(1-i)})\, A_{1\to n} = 0 \Rightarrow (y_i - q^{2(1-i)})(y_2 - q^{2})(y_3 - q^{2}) \cdots (y_n - q^{2}) = 0, \tag{4.3.14}$$

$(i = 2, \dots, n)$. Then two new types of idempotents (which are primitive orthogonal idempotents for the subalgebra $H_n \in M_{M+1}$) are obtained from these identities:

$$\begin{aligned} P\left(\begin{array}{|c|c|c|}\hline 1 & \dots & n\text{-}1 \\ \hline n \\ \cline{1-1}\end{array}\right) &= \frac{(y_n - q^{2(n-1)})}{(q^{-2} - q^{2(n-1)})} \prod_{k=1}^{n-1} \frac{(y_k - q^{-2})}{(q^{2(k-1)} - q^{-2})} = \\ &= S_{1\to n-1} - S_{1\to n} = \frac{[n-1]_q}{[n]_q}\, S_{1\to n-1}\, \sigma_{n-1}(q)\, S_{1\to n-1}, \end{aligned} \tag{4.3.15}$$

$$\begin{aligned} P\left(\begin{array}{|c|c|}\hline 1 & n \\ \hline \vdots \\ \cline{1-1} n\text{-}1 \\ \cline{1-1}\end{array}\right) &= \frac{(y_n - q^{2(1-n)})}{(q^{2} - q^{2(1-n)})} \prod_{k=1}^{n-1} \frac{(y_k - q^{2})}{(q^{2(1-k)} - q^{2})} = \\ &= A_{1\to n-1} - A_{1\to n} = \frac{[n-1]_q}{[n]_q}\, A_{1\to n-1}\, \sigma_{n-1}(q^{-1})\, A_{1\to n-1}, \end{aligned} \tag{4.3.16}$$

where [113, 200]

$$\sigma_n(x) := \lambda^{-1}\,(x^{-1}\sigma_n - x\sigma_n^{-1})$$

are Baxterized elements for the algebra $H_{M+1}(q)$ (the $R$-matrix representations of these elements are given in (3.8.5)). We consider properties of these elements below; see Eq. (4.3.38) and further discussion. The idempotents (4.3.15) and (4.3.16) are not central in $H_{M+1}$ but they are the elements of the commutative subalgebra $Y_{M+1}$.

*4.3.2. Primitive orthogonal idempotents in $H_{M+1}$ and Young tableaux*

Now we describe the general construction (see [208, 209] and references therein) of all primitive orthogonal idempotents $e_\alpha \in H_{M+1}$ which are elements of $Y_{M+1}$ (i.e., functions of the elements $y_i$). All these idempotents are common eigenidempotents of $y_i$:

$$y_i e_\alpha = e_\alpha y_i = a_i^{(\alpha)} e_\alpha \quad (i = 1, \dots, M+1),$$

where $a_i^{(\alpha)}$ are eigenvalues. We denote by $\mathrm{Spec}(y_1, \dots, y_{M+1})$ the set of strings of $(M+1)$ eigenvalues $\Lambda(e_\alpha) := (a_1^{(\alpha)}, \dots, a_{M+1}^{(\alpha)})$ $(\forall \alpha)$. The eigenidempotents $e_\alpha$ define left (and right) submodules $H_{M+1} \cdot e_\alpha$ (and $e_\alpha \cdot H_{M+1}$) in the regular bimodule of $H_{M+1}$.

**Lemma 1.** *The eigenidempotents $e$ and $e'$ with eigenvalues $a_i = a'_i$ $(\forall i = 1, \dots, M)$ and $a_{M+1} \neq a'_{M+1}$ define different left (right) submodules in the regular bimodule of $H_{M+1}$.*

**Proof.** We proof this Lemma only for the left submodules $H_{M+1} \cdot e$, $H_{M+1} \cdot e'$. The case of the right submodules can be considered analogously. Let $v$ and $v'$ be, respectively, elements of submodules $H_{M+1} \cdot e$ and $H_{M+1} \cdot e'$. Consider central element $Z = y_1 y_2 \cdots y_{M+1}$ (symmetric function of $y_i$). There are no elements $X \in H_{M+1}$ such that $v' = X\, v$, since the left action of $Z$ on elements $X\, v$ and $v'$ produces different eigenvalues. ∎

Now we introduce the important intertwining elements [84] (in another form these elements appeared in [204]):

$$U_{n+1} = \sigma_n y_n - y_n \sigma_n = \sigma_n y_n - \sigma_n^{-1} y_{n+1} = y_{n+1} \sigma_n^{-1} - y_n \sigma_n = \tag{4.3.17}$$

$$= (y_{n+1} - y_n)\sigma_n - \lambda y_{n+1} = \sigma_n (y_n - y_{n+1}) + \lambda y_{n+1} \;\; (1 \leqslant n \leqslant M), \tag{4.3.18}$$

subject to relations[20]

$$\begin{gathered} U_{n+1} y_n = y_{n+1} U_{n+1}, \quad U_{n+1} y_{n+1} = y_n U_{n+1}, \\ [U_{n+1},\, y_k] = 0 \;\; (k \neq n, n+1), \end{gathered} \tag{4.3.19}$$

$$U_n\, U_{n+1}\, U_n = U_{n+1}\, U_n\, U_{n+1}, \tag{4.3.20}$$

$$U_{n+1}^2 = (q y_n - q^{-1}\, y_{n+1})\,(q\, y_{n+1} - q^{-1}\, y_n). \tag{4.3.21}$$

**Lemma 2.** *The eigenidempotents $e$ and $e'$ with eigenvalues*

$$\begin{gathered} a_i = a'_i \;\; (\forall i = 1, \dots, M-1), \\ a_M = a'_{M+1}, \quad a_{M+1} = a'_M, \quad a_M \neq q^{\pm 2} a_{M+1} \end{gathered} \tag{4.3.22}$$

*belong to the same irreducible sub-bimodule in the regular bimodule of $H_{M+1}$.*

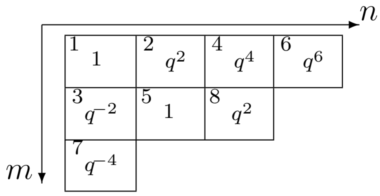
**Proof.** Since the algebra $Y_{M+1}$ generated by $\{y_1, \dots, y_{M+1}\}$ is maximal commutative subalgebra in $H_{M+1}$, we have $e' = e''$ if $\Lambda(e') = \Lambda(e'')$. Then, using intertwining element $U_{M+1}$ (4.3.17), we construct the eigenidempotent

$$e'' = \frac{1}{(q^2 a_M - a_{M+1})(a_{M+1} - q^{-2} a_M)}\, U_{M+1}\, e\, U_{M+1}, \qquad (e'')^2 = e'',$$

[20] The definition (4.3.17) of intertwining elements is not unique. One can multiply $U_{n+1}$ by a function $f(y_n, y_{n+1})$: $U_{n+1} \to U_{n+1} f(y_n, y_{n+1})$. Then Eqs. (4.3.19)–(4.3.21) are valid if $f$ satisfies $f(y_n, y_{n+1}) f(y_{n+1}, y_n) = 1$.

which is well defined in view of the last condition in (4.3.22). The element $U_{M+1}\, e\, U_{M+1}$ is not equal to zero, since $U_{M+1}^2\, e\, U_{M+1}^2 = (q^2 a_M - a_{M+1})^2 (a_{M+1} - q^{-2} a_M)^2 e \neq 0$. This inequality follows from the last condition in (4.3.22). For the element $e'' \sim U_{M+1}\, e\, U_{M+1}$ we have $\Lambda(e'') = \Lambda(e')$ in view of (4.3.19). Thus, $e'' = e' \Rightarrow e' \sim U_{M+1}\, e\, U_{M+1}$ and the eigenidempotents $e$ and $e'$ belong to the same irreducible sub-bimodule in the regular bimodule of $H_{M+1}$. ∎

Consider a Young diagram $[\nu]_{M+1}$ with $(M+1)$ nodes. We place the numbers $1, \dots, M+1$ into the nodes of the diagram in such a way that these numbers are arranged along rows and columns in ascending order in right and down directions. Such a diagram is called a standard Young tableau $\mathsf{T}_{[\nu]_{M+1}}$. Then we associate a number $q^{2(n-m)}$ (the "content") to each node of the standard Young tableau, where $(n, m)$ are coordinates of the node. Example:




(4.3.23)

In general, for the tableau $\mathsf{T}_{[\nu]_{M+1}}$, the $i$th node with coordinates $(n, m)$ looks like $\boxed{{}^{i}\, q^{2(n-m)}}$. Thus, to each standard Young tableau $[\nu]_n$ one can associate a string of numbers $\Lambda = (a_1, \dots, a_n)$ with $a_i = q^{2(n-m)}$. For example, a standard Young tableau (4.3.23) corresponds to a string

$$\Lambda = (1, q^2, q^{-2}, q^4, 1, q^6, q^{-4}, q^2).$$

Now we associate Young tableaux $\mathsf{T}_{[\nu]_{M+1}}$ (related to the primitive orthogonal idempotents) with paths in Young–Ogievetsky graph. By definition Young–Ogievetsky graph is a Young graph with vertices, which are Young diagrams, with edges, which indicate inclusions of diagrams (or a branching of representations), and with numbers (colours) on the edges corresponding to the eigenvalues of the Jucys–Murphy elements[21]. For example, the coloured Young–Ogievetsky graph for $H_4$ is

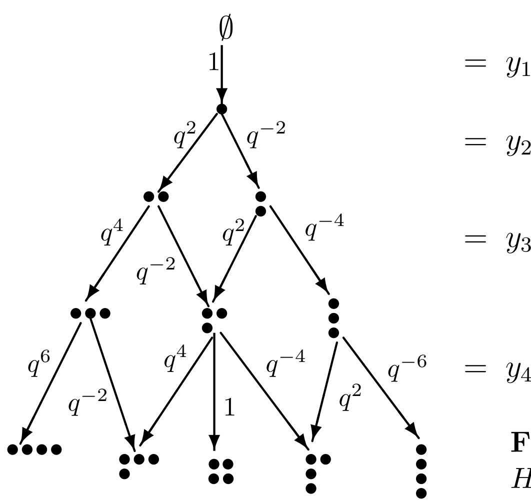



**Figure 4.1.** Young–Ogievetsky graph for $H_4(q)$.

[21]To our knowledge, O. Ogievetsky was the first who proposed to associate the eigenvalues of the Jucys–Murphy elements to the edges of the Young graph. Usually the indices on the edges of the graphs of Young type correspond to the multiplicity of the branching. In this case, the Young graph is called the Bratelli diagram. In our case, all multiplicities are equal to 1.

The paths (associated to Young tableaux) start from the top vertex $\emptyset$ and finish at the vertex labeled by the Young diagram of the same shape as the tableaux. The dimension of the corresponding representation of the Hecke algebra is equal to the number of standard tableaux of this shape or, as we see, the number of paths which lead to this Young diagram from top $\emptyset$. For example, the path $\{\emptyset \overset{1}{\to} \bullet \overset{q^2}{\to} \bullet\bullet \overset{q^{-2}}{\to} \begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix} \overset{1}{\to} \begin{smallmatrix}\bullet&\bullet\\ \bullet&\bullet\end{smallmatrix}\}$ corresponds to the tableau $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$, i.e., the shape of the tableau (Young diagram) is given by the shape of the last vertex of the path, while the numbers in nodes of the tableau show in which sequence the points $\bullet$ appear in the vertices along the path. The edge colours of the path (or contents of the nodes of the standard tableau, as it is explained in (4.3.23)) are the eigenvalues of the Jucys–Murphy elements $y_1 = 1$, $y_2 = q^2$, $y_3 = q^{-2}$, $y_4 = 1$ obtained by their action on the idempotent $P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right)$. Then the explicit formula for this idempotent can be constructed by induction. Namely, we take the explicit form of the previous idempotent $P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)$ (related to the previous vertex of the path) and multiply it by the factors $(y_4-1)$, $(y_4-q^4)$, and $(y_4-q^{-4})$, which correspond to possible colours of outgoing edges from vertex $\begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix}$, to obtain characteristic identity

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)(y_4-1)(y_4-q^4)(y_4-q^{-4}) = 0. \tag{4.3.24}$$

Then, to forbid the moving from the vertex $\begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix}$ along the edges with labels $q^4$ and $q^{-4}$ and move along the edge with the index 1 to the vertex $\begin{smallmatrix}\bullet&\bullet\\ \bullet&\bullet\end{smallmatrix}$, we remove from the left-hand side of (4.3.24) the factor $(y_4-1)$. As a result, we obtain (after an obvious renormalization)

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right) = P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)\frac{(y_4-q^4)(y_4-q^{-4})}{(1-q^4)(1-q^{-4})}. \tag{4.3.25}$$

In the same way, one can deduce the chain of identities

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) = P\left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right)\frac{(y_3-q^4)}{(q^{-2}-q^4)} = P\left(\begin{array}{|c|}\hline 1\\\hline\end{array}\right)\frac{(y_2-q^{-2})(y_3-q^4)}{(q^2-q^{-2})(q^{-2}-q^4)}, \tag{4.3.26}$$

where we fix $P\left(\begin{array}{|c|}\hline 1\\\hline\end{array}\right) = 1$ by definition. Using (4.3.26), the final formula for (4.3.25) can be written as

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right) = \frac{(y_2-q^{-2})(y_3-q^4)}{(q^2-q^{-2})(q^{-2}-q^4)}\frac{(y_4-q^4)(y_4-q^{-4})}{(1-q^4)(1-q^{-4})}. \tag{4.3.27}$$

We note that the described procedure leads automatically to the idempotents which are orthogonal to each other.

This example has demonstrated that all information about primitive orthogonal idempotents for the $A$-type Hecke algebra is encoded in the Young–Ogievetsky (YO) graph given in Figure 4.1. Thus, we need to justify this graph and its edge colours. First of all, we prove the following statement.

**Proposition 4.19.** *The spectrum of the Jucys–Murphy operators $y_j$ (possible edge indices of the YO graph) for $H_{M+1}$ is such that*

$$\mathrm{Spec}(y_j) \subset \{q^{2\mathbf{Z}_j}\} \quad \forall j = 1, 2, \ldots, M+1, \tag{4.3.28}$$

*where $\mathbf{Z}_j$ denotes the set of integer numbers $\{1-j, \ldots, -2, -1, 0, 1, 2, \ldots, j-1\}$.*

**Proof.** We use the important intertwining elements (4.3.17), (4.3.18) and prove (4.3.28) by induction. From the Hecke condition (4.3.1) we have

$$(y_2 - q^2)(y_2 - q^{-2}) = 0. \tag{4.3.29}$$

Thus, Spec($y_2$) satisfies (4.3.28). Assume that the spectrum of $y_{j-1}$ satisfies (4.3.28) for some $j \geqslant 3$. Consider a characteristic equation for $y_{j-1}$ (cf. (4.3.29)):

$$f(y_{j-1}) := \prod_\alpha (y_{j-1} - a^{(\alpha)}_{j-1}) = 0 \quad (a^{(\alpha)}_{j-1} \in \mathrm{Spec}(y_{j-1})).$$

Using operators $U_j$ and their properties (4.3.19), (4.3.21), we deduce

$$0 = U_j f(y_{j-1}) U_j = f(y_j) U_j^2 = f(y_j)(q^2 y_{j-1} - y_j)(y_j - q^{-2} y_{j-1}), \tag{4.3.30}$$

which means that

$$\mathrm{Spec}(y_j) \subset \big(\mathrm{Spec}(y_{j-1}) \cup q^{\pm 2} \cdot \mathrm{Spec}(y_{j-1})\big)\,, \tag{4.3.31}$$

and it justifies (4.3.28). ■

*4.3.3. Irreducible representations of $H_{M+1}$ and recurrence formula for primitive idempotents*

In [231], A. Okounkov and A. Vershik developed new approach to the construction of the irreducible representations of symmetric group (we review this approach in [139], Section 4.6). Here we generalize (following [201], [202–205], [208, 209]) the Okounkov–Vershik approach to the case of the Hecke algebra.

Consider a subalgebra $\hat{H}^{(i)}_2$ in $H_{M+1}$ with generators $y_i$, $y_{i+1}$ and $\sigma_i$ (for fixed $i \leqslant M$). We investigate (see [208, 209]) representations of $\hat{H}^{(i)}_2$ in the case when the elements $y_i$, $y_{i+1}$ are diagonalizable. Let $e$ be a common eigenidempotent of $y_i$, $y_{i+1}$: $y_i e = a_i e$, $y_{i+1} e = a_{i+1} e$. Then the left action of $\hat{H}^{(i)}_2$ closes on elements $v_1 = e$, $v_2 = \sigma_i e$ and is given by matrices

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}, \; y_i = \begin{pmatrix} a_i & -\lambda a_{i+1} \\ 0 & a_{i+1} \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} a_{i+1} & \lambda a_{i+1} \\ 0 & a_i \end{pmatrix}, \tag{4.3.32}$$

where we have used the standard convention $y v_i = v_j\, y_{ji}$ to produce matrix representations $||y_{ji}||$ for operators $y$.

The operators $y_i$, $y_{i+1}$ (4.3.32) can be simultaneously diagonalized by the transformation $y \to V^{-1} y V$, where

$$V = \begin{pmatrix} 1 & \frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -\frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}.$$

As a result, we obtain the following matrix representation:

$$\sigma_i = \begin{pmatrix} -\frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} & \frac{(a_i - q^2\, a_{i+1})(a_i - q^{-2}\, a_{i+1})}{(a_i - a_{i+1})^2} \\ 1 & \frac{\lambda\, a_i}{a_i - a_{i+1}} \end{pmatrix}, \; y_i = \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix}, \tag{4.3.33}$$

where $a_i \neq a_{i+1}$, otherwise $y_i$, $y_{i+1}$ are not diagonalizable. We note that the form of matrix $\sigma_i$ in (4.3.33) is not unique, since one can multiply $V$ by any diagonal matrix $D$ from the right.

For $a_{i+1} \neq q^{\pm 2} a_i$ one can perform additional similarity transformation of operators (4.3.33) with diagonal matrix $D = \mathrm{diag}(d_1, d_2)$ to make the matrix $\sigma_i$ symmetric:

$$\sigma_i \to D^{-1}\, \sigma_i\, D = \begin{pmatrix} -\frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} & \frac{d_1}{d_2} \\ \frac{d_1}{d_2} & \frac{\lambda\, a_i}{a_i - a_{i+1}} \end{pmatrix}, \quad \frac{d_1}{d_2} = \pm \frac{\sqrt{(a_i - q^2\, a_{i+1})(a_i - q^{-2}\, a_{i+1})}}{(a_i - a_{i+1})}.$$

When $a_{i+1} = q^{\pm 2} a_i$, the 2-dimensional representation (4.3.33) is reduced into the 1-dimensional representation with $\sigma_i = \pm q^{\pm 1}$, respectively. We summarize the above results as following.

**Proposition 4.20** *(q-Vershik–Okounkov [208, 209, 231]). Let*

$$\Lambda = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in \mathrm{Spec}(y_1, \dots, y_{M+1})$$

*be a possible spectrum of the commutative set $(y_1, \dots, y_{M+1})$, which corresponds to a primitive eigenidempotent $e_\Lambda \in H_{M+1}$. Then $a_i = q^{2m_i}$, where $m_i \in \mathbf{Z}_i$ (4.3.28) and*
*(1) $a_i \neq a_{i+1}$ for all $i < M+1$;*
*(2) if $a_{i+1} = q^{\pm 2} a_i$, then $\sigma_i \cdot e_\Lambda = \pm q^{\pm 1} e_\Lambda$;*
*(3) if $a_i \neq q^{\pm 2} a_{i+1}$, then*

$$\Lambda' = (a_1, \dots, a_{i+1}, a_i, \dots, a_{M+1}) \in \mathrm{Spec}(y_1, \dots, y_{M+1}),$$

*and the left action of the elements $\sigma_i, y_i, y_{i+1}$ in the linear span of $v_\Lambda = e_\Lambda$ and*

$$v_{\Lambda'} = \sigma_i\, e_\Lambda + \frac{\lambda a_{i+1}}{a_i - a_{i+1}}\, e_\Lambda$$

*is given by (4.3.33).*

From this Proposition we conclude that the only admissible subgraphs in the Young–Ogievetsky (YO) graph (subgraphs which show all possible two-edges paths with fixed initial and final vertices) are

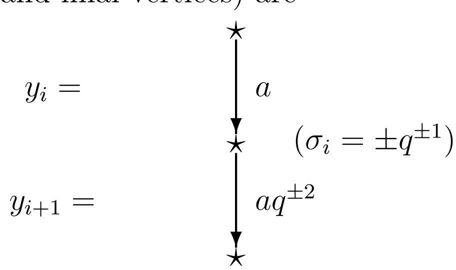



**Figure 4.2**

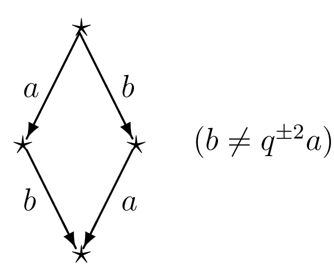



**Figure 4.3**

where stars in the vertices denote Young diagrams. These subgraphs are related to the 1-dimensional and 2-dimensional (it corresponds to the number of paths from the top vertex to the bottom one in Figure 4.2 and Figure 4.3) representations of the subalgebra generated by $\{y_i, y_{i+1}, \sigma_i\}$. In view of the braid relations $\sigma_i \sigma_{i\pm 1} \sigma_i = \sigma_{i\pm 1} \sigma_i \sigma_{i\pm 1}$ and possible values of $\sigma$'s presented in Figure 4.2 for 1-dimensional (1D) representation subgraphs, we conclude that the chains $\star \overset{a}{\longrightarrow} \star \overset{q^{\pm 2} a}{\longrightarrow} \star \overset{a}{\longrightarrow} \star$ of the 1D representation subgraphs in the YO graph in Figure 4.1 are forbidden. While admissible chains of 1D representation subgraphs are $\star \overset{a}{\longrightarrow} \star \overset{q^{\pm 2} a}{\longrightarrow} \star \overset{q^{\pm 4} a}{\longrightarrow} \star$. These statements and the form of only admissible subgraphs in Figures 4.2 and 4.3 justify (for

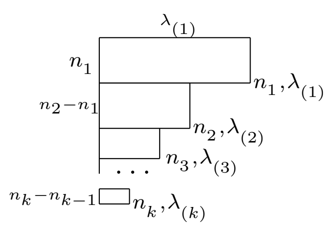
the $A$-type Hecke algebra) the YO graph presented in Figure 4.1. Indeed, we know the top of the YO graph which consists of 3 edges with indices $1, q^2, q^{-2}$ (see Figure 4.1). Then one can explicitly construct "step by step" moving down the whole YO graph (with all indices on edges) by using (4.3.31), the form of only admissible subgraphs in Figures 4.2, 4.3 and rules for the chains of 1-dimensional representations. We also stress here two important properties of the YO graph:
1) to each vertex of the graph the number of incoming edges $E_{\rm in}$ is less than the number of outgoing edges $E_{\rm out}$ on 1: $E_{\rm out} = E_{\rm in} + 1$;
2) to each vertex of the graph the products of indices $a_{\rm in}$ for incoming and $a_{\rm out}$ for outgoing edges are equal to each other: $\prod_{\alpha=1}^{E_{\rm in}} a_{\rm in}(\alpha) = \prod_{\beta=1}^{E_{\rm out}} a_{\rm out}(\beta)$.

Finally, we summarize all results about the spectrum $\mathrm{Spec}(y_1, \dots, y_{M+1})$ of the Jucys–Murphy elements $y_i \in H_{M+1}(q)$ and YO graph for the Hecke algebra $H_{M+1}(q)$ as following.

**Proposition 4.21** *([208, 209, 231]). There is a bijection between the set of the standard Young tableaux $\mathsf{T}_{[\nu]_n}$ with $n$ nodes, the set $\mathrm{Spec}(y_1, \dots, y_n)$ and the set of paths $X_{\vec{a}}$ in coloured YO graph which connect the top diagram $\emptyset$ and the diagram $[\nu]_n$.*

Since the YO graph is explicitly known, we can deduce the expressions (in terms of the elements $y_k$) of all orthogonal primitive idempotents for the Hecke algebra (in the same way as it has been done in (4.3.25)–(4.3.27)). We stress once again that the method of construction of explicit expressions for such primitive orthogonal idempotents is known and was discussed, e.g., in [202, 205, 208, 209]. Now we explain the operation of this method by using an arbitrary standard Young tableau as an example.

Let $\Lambda$ be a Young diagram with $n = n_k$ rows: $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $|\Lambda| := \sum_{i=1}^{n} \lambda_i$ be the number of its nodes. Consider the case when $\lambda_1 = \cdots = \lambda_{n_1} = \lambda_{(1)} > \lambda_{n_1+1} = \lambda_{n_1+2} = \cdots = \lambda_{n_2} = \lambda_{(2)} > \cdots > \lambda_{n_k - n_{k-1}+1} = \cdots = \lambda_{|\Lambda|} = \lambda_{(n_k)}$:

$\Lambda \;=$ 

(4.3.34)

Here $(n_i, \lambda_{(i)})$ are coordinates of the nodes corresponding to the inner corners of the diagram $\Lambda$. Consider any standard Young tableau $\mathsf{T}_{\Lambda_M}$ of shape (4.3.34) with $M = |\Lambda|$ nodes. Let $e(\mathsf{T}_{\Lambda_M}) \in H_M$ be a primitive idempotent corresponding to the tableau $\mathsf{T}_{\Lambda_M}$. Taking into account the branching rule implied by the coloured Young–Ogievetsky graph for $H_{M+1}$, we fix all possible eigenvalues $q^{2(\lambda_{(r)} - n_{r-1})}$ $(r = 1, \dots, k+1)$ of the element $y_{M+1}$. Then we conclude that the following identity holds:

$$e(\mathsf{T}_{\Lambda_M}) \prod_{r=1}^{k+1} \left( y_{M+1} - q^{2(\lambda_{(r)} - n_{r-1})} \right) = 0,$$

where $\lambda_{(k+1)} = n_0 = 0$. Thus, for a new tableau $\mathsf{T}_{\Lambda^j_{M+1}}$, which is obtained by adding to the tableau $\mathsf{T}_{\Lambda_M}$ of the shape (4.3.34) a new node with coordinates $(n_{j-1}+1, \lambda_{(j)}+1)$, we obtain the following primitive idempotent (after a normalization):

$$e(\mathsf{T}_{\Lambda^j_{M+1}}) := e(\mathsf{T}_{\Lambda_M}) \prod_{\substack{r=1 \\ r \neq j}}^{k+1} \frac{\left( y_{|\Lambda|+1} - q^{2(\lambda_{(r)} - n_{r-1})} \right)}{\left( q^{2(\lambda_{(j)} - n_{j-1})} - q^{2(\lambda_{(r)} - n_{r-1})} \right)} = e(\mathsf{T}_{\Lambda_M})\, \Pi_j \, . \qquad (4.3.35)$$

With the help of this formula and "initial data" $e\left(\boxed{1}\right) = 1$, one can deduce step by step explicit expressions for all primitive orthogonal idempotents for Hecke algebras.

**Remark.** The elements found by inductive formula (4.3.35) give by construction a complete system of mutually orthogonal idempotents in the Hecke algebra $H_M(q)$. Let $\mathsf{T}_{a,\Lambda}$ ($a = 1, \dots, f_\Lambda$) be standard Young tableaux of the shape of the Young diagram $\Lambda \vdash M$ and $f_\Lambda$ — the number of such Young tableaux of the shape $\Lambda$. A primitive idempotent $e(\mathsf{T}_{a,\Lambda}) \in H_M(q)$ corresponds to such tableau. Central idempotents $e(\Lambda)$ correspond to the Young diagrams $\Lambda \vdash M$ and are expressed as the sum $e(\Lambda) = \sum_{a=1}^{f_\Lambda} e(\mathsf{T}_{a,\Lambda})$. Then the completeness of the primitive orthogonal idempotents $e(\mathsf{T}_{a,\Lambda})$ is written as the resolution of unit operator 1 via central idempotents $e(\Lambda) \in H_M(q)$:

$$1 = \sum_{\Lambda \vdash M} \sum_{a=1}^{f_\Lambda} e(\mathsf{T}_{a,\Lambda}) = \sum_{\Lambda \vdash M} e(\Lambda).$$

One can find explicit formula for the number $f_\Lambda$ in [139], Subsection 4.3.2 (see also references therein).

*4.3.4. Idempotents in $H_{M+1}(q)$ and Baxterized elements. Matrix units in $H_{M+1}$*

Another convenient recurrent relations for Hecke symmetrizers and antisymmetrizers (4.3.5), (4.3.9), (4.3.10) are (see, e.g., [111, 113, 232]):

$$S_{1\to n} = S_{1\to n-1} \, \frac{\sigma_{n-1}(q^{1-n})}{[n]_q} \, S_{1\to n-1}, \quad S_{1\to n} = S_{2\to n} \, \frac{\sigma_1(q^{1-n})}{[n]_q} \, S_{2\to n}, \tag{4.3.36}$$

$$A_{1\to n} = A_{1\to n-1} \, \frac{\sigma_{n-1}(q^{n-1})}{[n]_q} \, A_{1\to n-1}, \quad A_{1\to n} = A_{2\to n} \, \frac{\sigma_1(q^{n-1})}{[n]_q} \, A_{2\to n}, \tag{4.3.37}$$

where $\sigma_n(x)$ are Baxterized elements [113, 200] (cf. (3.8.5)):

$$\sigma_n(x) := \lambda^{-1} \, (x^{-1}\sigma_n - x\sigma_n^{-1}), \tag{4.3.38}$$

for the algebra $H_{M+1}(q)$. We have already used these elements in definitions of the idempotents (4.3.15) and (4.3.16). The elements $\sigma_n(x)$ obey the Yang–Baxter equation (the proof of this statement is the same as in (3.8.1)–(3.8.4)):

$$\sigma_n(x) \, \sigma_{n-1}(xy) \, \sigma_n(y) = \sigma_{n-1}(y) \, \sigma_n(xy) \, \sigma_{n-1}(x). \tag{4.3.39}$$

These elements are also normalized by the conditions $\sigma_n(\pm 1) = \pm 1$ and satisfy

$$\sigma_i(x) = \frac{x - x^{-1}}{y - y^{-1}} \, \sigma_i(y) + \frac{yx^{-1} - xy^{-1}}{y - y^{-1}}, \qquad \forall x, y \neq \pm 1,$$

$$\sigma_i(x) \, \sigma_i(y) = \sigma_i(xy) + (x - x^{-1})(y - y^{-1})\lambda^{-2}. \tag{4.3.40}$$

The special case $y = x^{-1}$ of (4.3.40) gives the "unitarity condition"

$$\sigma_i(x) \, \sigma_i(x^{-1}) = \left(1 - \frac{(x - x^{-1})^2}{\lambda^2}\right) \equiv \frac{(qx^{-1} - q^{-1}x)(qx - q^{-1}x^{-1})}{\lambda^2}. \tag{4.3.41}$$

One can write the Baxterized elements (4.3.38) as rational function of $\sigma_i$ (cf. (3.8.6)):

$$\sigma_i(x) = \left(\frac{a^{-1}x - ax^{-1}}{\lambda x^2}\right) \frac{\sigma_i - ax^2}{\sigma_i - ax^{-2}}, \qquad \sigma_i^{(a)}(x) \; := \; \frac{\sigma_i - ax^2}{\sigma_i - ax^{-2}}, \tag{4.3.42}$$

where $a = -q$, or $a = q^{-1}$, and it becomes clear, for both choices of $a$, that the normalized elements $\sigma_i^{(a)}(x)$ obey $\sigma_i^{(a)}(x)\sigma_i^{(a)}(x^{-1}) = 1$.

Equations (4.3.40), (4.3.41), and (4.3.42) follow from the Hecke relation (4.3.1). The equivalence of both representations for (anti)symmetrizers given in the first and second equations of (4.3.36) and (4.3.37) is demonstrated by means of the Yang–Baxter equation (4.3.39), or by means of the obvious mirror automorphism $\sigma_k \to \sigma_{M-k}$ for the Hecke algebra $H_M(q)$. The equivalence of (4.3.36), (4.3.37), and (4.3.5) can be easily demonstrated if one writes the first representations of (4.3.36), (4.3.37) in the form

$$\begin{aligned} S_{1\to n} &= \tfrac{1}{[n]_q!}\, \sigma_1(q^{-1})\, \sigma_2(q^{-2}) \dots \sigma_{n-1}(q^{1-n})\, S_{1\to n-1}, \\ A_{1\to n} &= \tfrac{1}{[n]_q!}\, \sigma_1(q)\, \sigma_2(q^{2}) \dots \sigma_{n-1}(q^{n-1})\, A_{1\to n-1}, \end{aligned} \tag{4.3.43}$$

and then we should use (4.3.6) to compare (4.3.43) with (4.3.5) and (4.2.1). According to (4.3.36), (4.3.37), the first two projectors are (cf. (3.4.22), (4.3.7)):

$$P\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\right) = S_{12} = \frac{1}{[2]_q}\, \sigma_1(q^{-1}), \quad P\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right) = A_{12} = \frac{1}{[2]_q}\, \sigma_1(q),$$

and their orthogonality readily follows from the Hecke condition (4.3.1), or from (4.3.41). One can also express another types of the orthogonal idempotents (not only symmetrizers and antisymmetrizers) in terms of the Baxterized elements:

$$P\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \frac{1}{[3]_q!}\sigma_1(q^{-1})\sigma_2(q)\sigma_1(q^{-1}), \quad P\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = \frac{1}{[3]_q!}\sigma_1(q)\sigma_2(q^{-1})\sigma_1(q),$$

$$P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right) \sim \sigma_1(q^{-1})\sigma_2(q^{-2})\sigma_1(q^{-1})\sigma_3(q)\sigma_2(q^{-2})\sigma_1(q^{-1}),$$

$$P\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right) \sim \sigma_1(q)\sigma_2(q^{-1})\sigma_3(q^{-2})\sigma_2(q^{-1})\sigma_1(q),$$

$$P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\right) \sim \sigma_1(q^{-1})\sigma_2(q)\sigma_1(q^{-1})\sigma_3(q)\sigma_2(q^{-2})\sigma_1(q^{-1})\sigma_3(q^{-3}),$$

$$P\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right) \sim \sigma_1(q^{-1})\sigma_2(q)\sigma_1(q^{-1})\sigma_3(q^{3})\sigma_2(q)\sigma_1(q^{-1})\sigma_3(q^{-1}).$$

The method of presentation of all primitive orthogonal idempotents for the Hecke algebra in terms of the Baxterized elements was developed in [230] (see also references therein) by means of the fusion procedure.

**Remark 1.** Consider the quotients of the Hecke algebra $H_{M+1}(q)$ with respect to the additional relations $A_{1\to n} = 0$ $(n \leqslant M+1)$, which are equivalent (see (4.3.37)) to the identities

$$A_{1\to n-1}\, \sigma_{n-1}\, A_{1\to n-1} = \frac{q^{n-1}}{[n-1]_q}\, A_{1\to n-1}. \tag{4.3.44}$$

This is the way how the generalized Temperley–Lieb–Martin algebras [206] are defined. As it was mentioned in [113], the quotient of $H_{M+1}(q)$ with respect to the identity $A_{1\to 3} = 0$ is isomorphic to the Temperley–Lieb algebra.

**Remark 2.** By using intertwining elements (4.3.17) and Baxterized elements (4.3.38), one can immediately construct off-diagonal matrix units[22] (see Subsection **4.3.1**) in a double-sided Peirce decomposition of the Hecke algebra $H_{M+1} = \bigoplus_{\alpha,\beta} e_\alpha H_{M+1} e_\beta$. Let $P(X_{\vec{a}}) := e(\mathsf{T}_{\Lambda_{M+1}})$ be orthogonal primitive idempotent which corresponds to the Young tableau $\mathsf{T}_{\Lambda_{M+1}}$ or to the path $X_{\vec{a}}$ on the coloured Young–Ogievetsky graph labeled by the eigenvalues

$$\begin{gathered} \vec{a} = (1, a_2, \dots, a_{M+1}) \in \mathrm{Spec}(y_1, \dots, y_{M+1}), \\ y_i\, P(X_{\vec{a}}) = P(X_{\vec{a}})\, y_i = a_i\, P(X_{\vec{a}}) \;\; (\forall i = 1, \dots, M+1). \end{gathered} \tag{4.3.45}$$

In the case $a_j \neq q^{\pm 2} a_{j+1}$ (see the proof of Lemma 2 in Subsection **4.3.2**), we introduce the element $P(X_{s_j \cdot \vec{a}}) \in H_{M+1}$:

$$P(X_{s_j\cdot\vec{a}}) := \frac{1}{(q^2 a_j - a_{j+1})(a_{j+1} - q^{-2} a_j)}\, U_{j+1}\, P(X_{\vec{a}})\, U_{j+1} \quad (\forall j = 1, \dots, M) \tag{4.3.46}$$

such that

$$\begin{gathered} P(X_{s_j\cdot\vec{a}})^2 = P(X_{s_j\cdot\vec{a}}), \quad \vec{y}\, P(X_{s_j\cdot\vec{a}}) = P(X_{s_j\cdot\vec{a}})\, \vec{y} = (s_j \cdot \vec{a}) P(X_{s_j\cdot\vec{a}}), \\ P(X_{s_j\cdot\vec{a}}) = \frac{(q^2 y_j - y_{j+1})(y_{j+1} - q^{-2} y_j)}{(q^2 a_j - a_{j+1})(a_{j+1} - q^{-2} a_j)}\, (s_j \cdot P)(X_{\vec{a}}), \\ P(X_{s_j\cdot\vec{a}})\, P(X_{\vec{a}}) = P(X_{\vec{a}})\, P(X_{s_j\cdot\vec{a}}) = 0, \end{gathered} \tag{4.3.47}$$

where $s_j \cdot \vec{a} = (a_1, \dots, a_{j+1}, a_j, \dots a_{M+1}) \in \mathrm{Spec}(y_1, \dots, y_{M+1})$ is the vector with permuted coordinates $a_j$ and $a_{j+1}$; $(s_j \cdot P)(X_{\vec{a}})$ denotes the function $P(X_{\vec{a}})$ with permuted variables $y_i$ and $y_{i+1}$. The identity (4.3.47) follows from the fact that $P := P(X_{\vec{a}})\; P(X_{\vec{a}\,'}) \;= 0$ for all $\vec{a} \neq \vec{a}\,'$ (i.e., $\exists\, j$: $a_j \neq a'_j$) in view of the equations $y_j P = a_j P = a'_j P$ which follow from $y_j\, P(X_{\vec{a}}) = P(X_{\vec{a}})\, y_j$.

According to (4.3.21) and (4.3.46), we obtain

$$\begin{gathered} U_{j+1}\, P(X_{\vec{a}}) = P(X_{s_j\cdot\vec{a}})\, U_{j+1} =: P(X_{s_j\cdot\vec{a}} | X_{\vec{a}}) \;\; (j = 1, \dots, M), \\ P(X_{\vec{a}})\, U_{j+1} = U_{j+1}\, P(X_{s_j\cdot\vec{a}}) =: P(X_{\vec{a}} | X_{s_j\cdot\vec{a}}) \;\; (j = 1, \dots, M). \end{gathered} \tag{4.3.48}$$

In the case $a_j \neq q^{\pm 2} a_{j+1}$, in view of Lemma 2, we have $s_j \cdot \vec{a} \in \mathrm{Spec}(y_1, \dots, y_{M+1})$ (the path $X_{s_j\cdot\vec{a}}$ exists in the Young–Ogievetsky graph and corresponds to the standard Young tableau). Then, taking into account (4.3.17), (4.3.18), we deduce

$$\left.\begin{aligned} \sigma_j(a_j, a_{j+1})\; P(X_{\vec{a}}) = -P(X_{s_j\cdot\vec{a}})\; \sigma_j(a_{j+1}, a_j) = P(X_{s_j\cdot\vec{a}} | X_{\vec{a}}), \\ -P(X_{\vec{a}})\; \sigma_j(a_j, a_{j+1}) = \sigma_j(a_{j+1}, a_j)\; P(X_{s_j\cdot\vec{a}}) = P(X_{\vec{a}} | X_{s_j\cdot\vec{a}}), \end{aligned}\right\} \;\; \Rightarrow \tag{4.3.49}$$

$$\begin{gathered} (\sigma_j + \tfrac{\lambda a_{j+1}}{(a_j - a_{j+1})})\, P(X_{\vec{a}}) = P(X_{s_j\cdot\vec{a}})\, (\sigma_j - \tfrac{\lambda a_j}{(a_j - a_{j+1})}) = \tfrac{P(X_{s_j\cdot\vec{a}} | X_{\vec{a}})}{(a_j - a_{j+1})}, \\ P(X_{\vec{a}})\, (\sigma_j + \tfrac{\lambda a_{j+1}}{(a_j - a_{j+1})}) = (\sigma_j - \tfrac{\lambda a_j}{(a_j - a_{j+1})})\, P(X_{s_j\cdot\vec{a}}) = \tfrac{P(X_{\vec{a}} | X_{s_j\cdot\vec{a}})}{(a_{j+1} - a_j)}, \end{gathered} \tag{4.3.50}$$

where we used the Baxterized elements (cf. (4.3.38))

$$\sigma_n(x, y) = x\, \sigma_n - y\, \sigma_n^{-1} = (x - y)\, \sigma_n + y\, \lambda \tag{4.3.51}$$

[22] Recall that the orthogonal primitive idempotents are diagonal matrix units.

subject to the Yang–Baxter equation (4.3.39) written in the form

$$\sigma_n(x,y)\,\sigma_{n+1}(x,z)\,\sigma_n(y,z) = \sigma_{n+1}(y,z)\,\sigma_n(x,z)\,\sigma_{n+1}(x,y).$$

It was shown in [205] that the elements $P(X_{s_j\cdot\vec{a}}|X_{\vec{a}})$ play the role of the off-diagonal matrix elements in the Peirce decomposition (see Subsection **4.3.1**). In the case $a_j = q^{\pm 2}a_{j+1}$, we have

$$\begin{gathered} U_{j+1}\,P(X_{\vec{a}}) = 0 = P(X_{\vec{a}})\,U_{j+1} \quad \Rightarrow \\ (\sigma_j \pm q^{\mp 1})\,P(X_{\vec{a}}) = 0 = -P(X_{\vec{a}})\,(\sigma_j \pm q^{\mp 1}), \end{gathered} \tag{4.3.52}$$

where the second line in (4.3.52) follows from (4.3.50) and defines the 1-dimensional representation for the generator $\sigma_j$ corresponding to Figure 4.2 in Subsection **4.3.3**.

Since the Hecke algebras $H_{M+1}$ are semisimple, we have the following identity:

$$P(X_{\vec{a}})\,B\,P(X_{\vec{a}}) = C_{\vec{a}}(B)\,P(X_{\vec{a}}), \qquad \forall B \in H_{M+1}, \tag{4.3.53}$$

where $P(X_{\vec{a}})$ is any primitive idempotent in $H_{M+1}$ and $C_{\vec{a}}(B)$ is a constant which depends on the element $B$ and the path $X_{\vec{a}}$ in the coloured Young–Ogievetsky graph (i.e., it depends on the vector $\vec{a}$ from the spectrum (4.3.45)). To justify identity (4.3.53), we check it for any monomial $B = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_r}$ in generators $\sigma_i \in H_{M+1}$ of any order $r$. We require that the monomial $B$ can not be reduced to the polynomial of order less than $r$ by means of relations (4.1.1) and (4.3.1). Then we use the induction to prove (4.3.53). We note that by using the definition (4.3.17) of $U_{i+1}$ and then (4.3.18), we obtain the base of the induction:

$$\begin{gathered} 0 = P(X_{\vec{a}})\,U_{i+1}\,P(X_{\vec{a}}) = P(X_{\vec{a}})\,((y_{i+1}-y_i)\sigma_i - \lambda y_{i+1})\,P(X_{\vec{a}}) = \\ = (a_{i+1}-a_i)P(X_{\vec{a}})\,\sigma_i\,P(X_{\vec{a}}) - \lambda a_{i+1}\,P(X_{\vec{a}}) \quad \Rightarrow \quad P(X_{\vec{a}})\,\sigma_i\,P(X_{\vec{a}}) = \frac{\lambda a_{i+1}}{(a_{i+1}-a_i)}P(X_{\vec{a}}). \end{gathered}$$

Let the identity (4.3.53) be correct for all monomials $B = \sigma_{i_1}\cdots\sigma_{i_k}$ when $k \leqslant r$. We need to prove (4.3.53) for monomials $B = \sigma_{i_1}\cdots\sigma_{i_{r+1}}$ of order $(r+1)$. Consider the element $P(X_{\vec{a}})\,U_{i_1+1}\cdots U_{i_{r+1}+1}\,P(X_{\vec{a}})$ and start to commute left idempotent $P(X_{\vec{a}})$ to the right with the help of (4.3.48). We have two possibilities.

**1.** The first one is

$$\begin{gathered} P(X_{\vec{a}})\,U_{i_1+1}\cdots U_{i_{r+1}+1}\,P(X_{\vec{a}}) = U_{i_1+1}\cdots U_{i_k+1}P(X_{\vec{a}^{(k)}})U_{i_{k+1}+1}\cdots\,P(X_{\vec{a}}) = \\ = \cdots = U_{i_1+1}\cdot\cdots\cdot U_{i_{r+1}+1}\,P(X_{\vec{a}^{(r+1)}})\;P(X_{\vec{a}}) = 0, \end{gathered} \tag{4.3.54}$$

$$\vec{a}^{(k)} := s_{i_k}\cdot\vec{a}^{(k-1)}, \quad \vec{a}^{(0)} := \vec{a}, \quad s_i\cdot(v_1,\dots,v_i,v_{i+1},\dots) = (v_1,\dots,v_{i+1},v_i,\dots.),$$

where $(\vec{a}^{(k)})_{i_{k+1}} \neq q^{\pm 2}(\vec{a}^{(k)})_{i_{k+1}+1}$ $(\forall k = 0,\dots,r)$ and we used the orthogonality $P(X_{\vec{a}'})\cdot P(X_{\vec{a}}) = 0$ for $\vec{a}' \neq \vec{a}$ in the last equality in (4.3.54). In this case, by using (4.3.49), we deduce

$$\begin{gathered} 0 = P(X_{\vec{a}})\,U_{i_1+1}\cdots U_{i_{r+1}+1}\,P(X_{\vec{a}}) = U_{i_1+1}\cdots U_{i_r+1}\,P(X_{\vec{a}^{(r)}})\,U_{i_{r+1}+1}\,P(X_{\vec{a}}) = \\ = (-1)\,U_{i_1+1}\cdots U_{i_{r-1}+1}\,P(X_{\vec{a}^{(r-1)}})\,U_{i_r}\sigma_{i_{r+1}}(a^{(r)}_{i_{r+1}}, a^{(r)}_{i_{r+1}+1})\,P(X_{\vec{a}}) = \cdots = \\ = (-1)^{r+1}P(X_{\vec{a}})\,\sigma_{i_1}(a^{(0)}_{i_1}, a^{(0)}_{i_1+1})\cdots\sigma_{i_r}(a^{(r-1)}_{i_r}, a^{(r-1)}_{i_r+1})\,\sigma_{i_{r+1}}(a^{(r)}_{i_{r+1}}, a^{(r)}_{i_{r+1}+1})\;\;P(X_{\vec{a}}), \end{gathered} \tag{4.3.55}$$

where $\sigma_i(x,y)$ are Baxterized elements (4.3.51). The substitution of the r.h.s. of (4.3.51) gives

$$P(X_{\vec{a}})\,\sigma_{i_1}\cdots\sigma_{i_{r+1}}\,P(X_{\vec{a}}) = \frac{1}{(a^{(0)}_{i_1+1}-a^{(0)}_{i_1})\cdots(a^{(r)}_{i_{r+1}+1}-a^{(r)}_{i_{r+1}})}P(X_{\vec{a}})\,\overline{B}\,P(X_{\vec{a}}), \tag{4.3.56}$$

where $\overline{B}$ is a polynomial in $\sigma_i \in H_{M+1}$ of order less than $(r+1)$ and, therefore, in view of the induction conjecture, we obtain (4.3.53).

**2.** The second possibility occurs if at some step $k$ in (4.3.54) the condition $(\vec{a}^{(k)})_{i_{k+1}} = q^{\pm 2}(\vec{a}^{(k)})_{i_{k+1}+1}$ arises. In this case, in view of (4.3.52), we obtain for any element $A \in H_{M+1}$:

$$P(X_{\vec{a}})\, U_{i_1+1} \cdots U_{i_{k+1}+1}\, A\, P(X_{\vec{a}}) = U_{i_1+1} \cdots U_{i_k+1} P(X_{\vec{a}^{(k)}}) U_{i_{k+1}+1}\, A\, P(X_{\vec{a}}) = 0. \qquad (4.3.57)$$

We take here $A = \sigma_{i_{k+1}+1} \cdots \sigma_{i_{r+1}}$ and write (4.3.57) with the help of (4.3.49) and (4.3.52) (in the same way as in (4.3.55)) in the form

$$0 = P(X_{\vec{a}})\, U_{i_1+1} \cdots U_{i_{k+1}+1}\, \sigma_{i_{k+1}+1} \cdots \sigma_{i_{r+1}}\, P(X_{\vec{a}}) =$$
$$= (-1)^{k-1} P(X_{\vec{a}})\, \sigma_{i_1}(a^{(0)}_{i_1}, a^{(0)}_{i_1+1}) \cdots \sigma_{i_{k-1}}(a^{(k-2)}_{i_{k-1}}, a^{(k-2)}_{i_{k-1}+1})(\sigma_{i_k} \pm q^{\mp 1})\, \sigma_{i_{k+1}+1} \cdots \sigma_{i_{r+1}}\, P(X_{\vec{a}}) \quad \Rightarrow$$

$$P(X_{\vec{a}})\, \sigma_{i_1} \cdots \sigma_{i_{r+1}}\, P(X_{\vec{a}}) = \frac{1}{(a^{(0)}_{i_1+1} - a^{(0)}_{i_1}) \cdots (a^{(k-1)}_{i_k+1} - a^{(k-1)}_{i_k})} P(X_{\vec{a}})\, \overline{B}_{\pm}\, P(X_{\vec{a}}), \qquad (4.3.58)$$

where $\overline{B}_{\pm}$ are polynomials in $\sigma_i \in H_{M+1}$ of order less than $(r+1)$ and, thus, in view of the induction conjecture, we again prove (4.3.53).

Finally, we note that equations (4.3.55), (4.3.56), and (4.3.58) give us the possibility to calculate explicitly the constant $C_{\vec{a}}(B)$ in (4.3.53).

### *4.3.5. Affine Hecke algebras and reflection equation*

**1.** In this subsection, we consider the infinite-dimensional Hecke algebra, which corresponds to the affine $A^{(1)}$-type Coxeter graph (4.1.8), with generators $\sigma_i$ $(i = 1, \ldots, M)$ subject to relations (4.3.1), (4.1.6). Thus, this algebra is the quotient of the algebra $\mathbb{C}[\mathcal{B}_{M+1}(A^{(1)})]$ with respect to additional Hecke relations (4.3.1). We call this algebra a periodic $A$-type Hecke algebra[23] and denote it as $AH_{M+1}$. For the algebra $AH_{M+1}$ one can construct the set of $(M-1)$ commuting elements

$$I_k = \sum_{i=1}^{M} \sigma_i \sigma_{i+1} \ldots \sigma_{i+k} \qquad (k = 0, \ldots, M-2), \qquad (4.3.59)$$

where we have identified $\sigma_{M+i} = \sigma_i$. The first element in (4.3.59) is $I_0 = \sum_{i=1}^{M} \sigma_i$ and, in the $R$-matrix representation, this element gives a Hamiltonian for periodic spin chain (see (5.1.14) in Subsection **5.1**).

Let $\{\sigma_1, \ldots, \sigma_{M-1}\}$ be generators of the braid group $\mathcal{B}_M$. We extend the group $\mathcal{B}_M$ by the element $X$ such that

$$X\, \sigma_k\, X^{-1} = \sigma_{k-1} \quad (\forall k = 2, \ldots, M-1), \qquad (4.3.60)$$
$$X\, \sigma_1\, X^{-1} = X^{-1}\, \sigma_{M-1}\, X =: \sigma_M. \qquad (4.3.61)$$

It is not hard to check that the new element $\sigma_M$ satisfies Eqs. (4.1.3) and, therefore, the elements $\{\sigma_1, \ldots, \sigma_M\}$ (where $\sigma_M$ has been defined in (4.3.61)) generate the periodic braid group $\overline{\mathcal{B}}_M = \mathcal{B}_M(A^{(1)})$.

Note that $X$ (4.3.60), (4.3.61) can be realized as the inner element of $\mathcal{B}_M \subset \mathcal{B}_M(A^{(1)})$. Indeed, the operator $X$ which solves Eqs. (4.3.60), (4.3.61) can be taken in the form $X =$

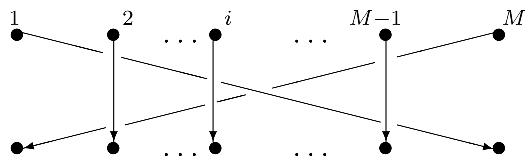
[23] As we will see below, in Subsection **5.1**, this algebra appears in a formulation of the integrable periodic spin chain models.

$\sigma_{M-1\leftarrow 1} \in \mathcal{B}_M$, where the notation $\sigma_{M-1\leftarrow m} := \sigma_{M-1} \dots \sigma_{m+1} \sigma_m$ has been used. Then we define the additional generator $\sigma_M$ (4.3.61) as

$$\sigma_M := X\, \sigma_1\, X^{-1} = \sigma_{M-1\leftarrow 1}\, \sigma_1\, \sigma^{-1}_{M-1\leftarrow 1} = \sigma_{M-1\leftarrow 1}\, \sigma^{-1}_{M-1\leftarrow 2} \tag{4.3.62}$$

and its graphical representation is

$\sigma_M$ =




(4.3.63)

It is evident that $\sigma_M$ satisfies (4.1.3) in view of its graphical representation (4.3.63). According to Eqs. (4.1.1) and (4.1.3), the elements $\{\sigma_1, \dots, \sigma_M\}$ of the group $\mathcal{B}_M$ (where $\sigma_M$ has been defined in (4.3.62)) generate the periodic braid group $\overline{\mathcal{B}}_M$ and, therefore, Eq. (4.3.62) defines the homomorphism: $\overline{\mathcal{B}}_M \to \mathcal{B}_M$.

**2.** Another infinite-dimensional Hecke algebra is an affine Hecke algebra $\hat{H}_{M+1}(q)$. We recall that the affine Hecke algebra $\hat{H}_{M+1}(q)$ is defined[24] (see, e.g., [207], Chapter 12.3) as algebra generated by elements $\sigma_i$ $(i = 1, \dots, M)$ of $H_{M+1}(q)$ and additional generators $y_k$ $(k = 1, \dots, M+1)$ subject to relations (cf. (4.1.11), (4.1.12)):

$$y_{k+1} = \sigma_k\, y_k\, \sigma_k, \quad y_k\, y_j = y_j\, y_k, \quad y_j\, \sigma_i = \sigma_i\, y_j \;\; (j \neq i, i+1) \tag{4.3.64}$$

(the generators $\{y_k\}$ form a commutative subalgebra in $\hat{H}_{M+1}(q)$). We note that, in view of the first relation in (4.3.64), the minimal set of generators of $\hat{H}_{M+1}(q)$ is $\{\sigma_1, \dots, \sigma_M, y_1\}$. Symmetric functions of the elements $y_k$ generate the center of the algebra $\hat{H}_{M+1}(q)$. Below, to be short, we omit the parameter $q$ in the notation $H_{M+1}(q)$ and $\hat{H}_{M+1}(q)$. The interesting property of the algebra $\hat{H}_{M+1}$ is the existence of the important intertwining elements [84] (cf. (4.3.17) and elements $\phi_i$ in [204], Proposition 3.1):

$$U_{i+1} = (\sigma_i y_i - y_i \sigma_i)\; f(y_i, y_{i+1}), \quad (1 \leqslant i \leqslant M),$$

where function $f$ satisfies: $f(y_i, y_{i+1}) f(y_{i+1}, y_i) = 1$. The elements $U_i$ obey the same relations (4.3.19)–(4.3.21) as in the case of the non-affine $A$-type Hecke algebra $H_{M+1}$.

Now we describe the procedure how one can construct $(M+1)$-dimensional representation for the Hecke subalgebra $H_{M+1} \subset \hat{H}_{M+1}$. Let $v$ be a vector in the space of 1-dimensional representation of $H_{M+1}$ such that $\sigma_i\, v = q\, v$ $(\forall i = 1, \dots, M)$. Consider the induced $(M+1)$-dimensional space with the basis $\{v_1, v_2, \dots, v_{M+1}\}$, where $v_k := y_k v$. Then, according to (4.3.64) and the Hecke condition (4.3.1), we obtain $(M+1)$-dimensional representation for generators $\sigma_i$:

$$\sigma_i\, v_k = q\, v_k \;\; (k \neq i, i+1), \quad \sigma_i\, v_i = q^{-1}\, v_{i+1}, \quad \sigma_i\, v_{i+1} = \lambda\, v_{i+1} + q\, v_i,$$

which is called the *Burau representation* of $H_{M+1}$. The matrix form of this representation is

$$\sigma_i = \mathrm{diag}\Big(\underbrace{q, \dots, q,}_{i-1} \begin{pmatrix} 0 & q \\ 1/q & \lambda \end{pmatrix}, \underbrace{q, \dots, q}_{M-i}\Big). \tag{4.3.65}$$

[24] This algebra is isomorphic to the quotient of the braid group algebra $\mathbb{C}[\mathcal{B}_{M+1}(C)]$, where the generators $\sigma_i \in \mathcal{B}_{M+1}(C)$ $(i = 1, \dots, M,\; i \neq 0)$ are constrained by additional Hecke conditions (4.3.1). The definition of $\mathcal{B}_{M+1}(C)$ is given in Subsection **4.1** and is related to the Coxeter graph (4.1.9).

One can start from another possible one-dimensional representation $\sigma_i\, v = -q^{-1}\, v$ $(\forall i)$ of $H_{M+1}$, which leads to a new Burau representation resulting from (4.3.65) by replacing $q \to (-1/q)$.

The affine Hecke algebra $\hat{H}_{M+1}$ makes it possible to formulate the universal Baxterized solution of the reflection equation (see (4.4.48), (5.2.3) below):

$$\sigma_n(x\, z^{-1})\, K_n(x)\, \sigma_n(x\, z)\, K_n(z) = K_n(z)\, \sigma_n(x\, z)\, K_n(x)\, \sigma_n(x\, z^{-1}), \tag{4.3.66}$$

where $x, z$ are spectral parameters and Baxterized elements $\sigma_n(x) \in H_{M+1}$ are defined in (4.3.38). The reflection equation (4.3.66) appears, e.g., in the theory of integrable spin chains with boundaries [263] and in 2D quantum integrable field theories [262] (see also Subsection **5.2** below). Taking the reflection operator $K_n(x)$ in the form

$$K_n(x) = \frac{y_n - \xi\, x^2}{y_n - \xi\, x^{-2}}, \tag{4.3.67}$$

where $\xi$ is any constant, we find [211] that this $K_n(x)$ is a solution of (4.3.66) if $y_n$ are the affine generators of $\hat{H}_{M+1}$. In particular, one can easily reduce (4.3.67) to the solution

$$K_n(x) = y_n + \frac{\beta_0/\xi + \xi + \beta_1 x^2}{x^2 - x^{-2}} \tag{4.3.68}$$

of the reflection equation (4.3.66) if, in addition, we require that $y_n$ satisfies a quadratic characteristic equation $y_n^2 + \beta_1 y_n + \beta_0 = 0$ $(\forall \beta_0, \beta_1 \in \mathbb{C}\backslash 0)$. The solution (4.3.67) is obviously regular: $K_n(1) = 1$, and obeys a "unitary condition":

$$K_n(x) K_n(x^{-1}) = 1.$$

We stress that the simplest solution (4.3.68) of the reflection equation (4.3.66) was considered in [217–219] (another special solutions were found in [220]).

If one has a solution of Eq. (4.3.66) for $n = m$, then a solution for $n = m + 1$ can be constructed by means of the formula

$$K_{m+1}(x) = (\lambda x)^2\, \sigma_m(x) K_m(x) \sigma_m(x).$$

In particular, one can take $K_{n-1}(x) = 1$ and, using (4.3.39) and (4.3.40), directly check that (cf. (4.3.68))

$$\frac{K_n(x)}{x^2} = \lambda^2\, \sigma_{n-1}^2(x) = \lambda^2 \left( \sigma_{n-1}(x^2) + \frac{(x - x^{-1})^2}{\lambda^2} \right) = \sigma_{n-1}^2 + \frac{2 - (2 + \lambda^2)\, x^2}{x^2 - x^{-2}}$$

solves Eq. (4.3.66).

**Remark 1.** Consider the following inclusions of the subalgebras $\hat{H}_1 \subset \hat{H}_2 \subset \cdots \subset \hat{H}_{M+1}$:

$$\{y_1; \sigma_1, \ldots, \sigma_{n-1}\} = \hat{H}_n \subset \hat{H}_{n+1} = \{y_1; \sigma_1, \ldots, \sigma_{n-1}, \sigma_n\}.$$

Then, following [211, 212], we equip the algebra $\hat{H}_{M+1}$ by linear mappings

$$\mathrm{Tr}_{D(n+1)} : \quad \hat{H}_{n+1} \to \hat{H}_n,$$

from the algebra $\hat{H}_{n+1}$ to its subalgebras $\hat{H}_n$, such that for all $X, X' \in \hat{H}_n$ and $Y \in \hat{H}_{n+1}$ we have

$$\begin{gathered} \mathrm{Tr}_{D(n+1)}(X) = Z^{(0)}\, X, \;\; \mathrm{Tr}_{D(n+1)}(X\, Y\, X') = X\, \mathrm{Tr}_{D(n+1)}(Y)\, X' \;\;, \\ \mathrm{Tr}_{D(n+1)}(\sigma_n^{\pm 1} X \sigma_n^{\mp 1}) = \mathrm{Tr}_{D(n)}(X), \quad \mathrm{Tr}_{D(n+1)}(X \sigma_n X') = X\, X', \\ \mathrm{Tr}_{D(1)}(y_1^k) = Z^{(k)}, \;\; \mathrm{Tr}_{D(n)} \mathrm{Tr}_{D(n+1)}(\sigma_n Y) = \mathrm{Tr}_{D(n)} \mathrm{Tr}_{D(n+1)}(Y \sigma_n), \end{gathered} \tag{4.3.69}$$

where $Z^{(k)} \in \mathbf{C}\backslash\{0\}$ ($k \in \mathbb{Z}$) are constants. We stress that $Z^{(k)}$ could be considered as additional generators of an Abelian subalgebra $\hat{H}_0$ which extends $\hat{H}_{M+1}$ and are central in $\hat{H}_{M+1}$, but for us it is enough to put $Z^{(k)}$ to constants.

Using the maps $\mathrm{Tr}_{D(n+1)}$, one can construct the elements

$$\tau_n(x) = \mathrm{Tr}_{_{D(n+1)}}\big(\sigma_n(x)\cdots\sigma_1(x)\,K_1(x)\,\sigma_1(x)\cdots\sigma_n(x)\big) \;\in\; \hat{H}_n, \tag{4.3.70}$$

where $\sigma_i(x)$ are Baxterized elements (4.3.38) and $K_1(x)$ is a solution (4.3.67) of the reflection equation (4.3.66) for $n = 1$. The elements (4.3.70) are generating functions for a commutative family of elements in $\hat{H}_n$, since we have (see [211, 212])

$$[\tau_n(x),\, \tau_n(z)] = 0 \quad (\forall x, z).$$

Moreover, the elements (4.3.70) are analogs of Sklyanin's transfer-matrices [263] and, making use of the elements $\tau_n(x)$, one can formulate [212] the integrable open Hecke chain models with nontrivial boundary conditions. These models generalize the quantum integrable spin models of the Heisenberg type. The local Hamiltonian of the open Hecke chain is

$$\mathcal{H}_n = \sum_{m=1}^{n-1} \sigma_m - \frac{\lambda}{2}\, y_1'(1). \tag{4.3.71}$$

This Hamiltonian (up to a normalization factor and additional constant) can be obtained by differentiating $\tau_n(x)$ with respect to spectral parameter $x$ at the point $x = 1$. The Hamiltonian (4.3.71) describes the open chain model with nontrivial boundary condition on the first site (given by the second term in (4.3.71)) and free boundary condition on the last site of the chain. In [212], we show that the transfer matrix elements $\tau_n(x)$ satisfy functional relations generalizing functional relations ($T - Q$ relations) for transfer matrices in solvable open spin chain models (see, e.g., [215, 216] and references therein).

**Remark 2.** Interrelations of periodic $AH_M$ (see point **1.** above) and affine $\hat{H}_M$ (see point **2.** above) Hecke algebras has been discussed in [205]. Here we present more explicit construction [210] of these interrelations which is valid even for the braid group case (when the Hecke condition (4.3.1) is relaxed).

Consider the affine braid group $\hat{\mathcal{B}}_M = \mathcal{B}_M(C)$ (see Definition **14** in Subsection **4.1**) with generators $\{\sigma_1, \ldots, \sigma_{M-1}, y_1\}$. The generator $y_1$ satisfies reflection equation and locality conditions

$$\sigma_1\, y_1\, \sigma_1\, y_1 = y_1\, \sigma_1\, y_1\, \sigma_1, \quad [y_1,\, \sigma_k] = 0 \;\; (k = 2, \ldots, M-1).$$

Then the operator

$$X = \sigma_{M-1\leftarrow 1}\, y_1 \in \hat{\mathcal{B}}_M$$

solves Eqs. (4.3.60), (4.3.61) and one can introduce new generator $\sigma_M \in \hat{\mathcal{B}}_M$ according to (4.3.61):

$$\sigma_M = \sigma_{M-1\leftarrow 1}\, y_1\, \sigma_1\, y_1^{-1}\, \sigma_{M-1\leftarrow 1}^{-1}, \tag{4.3.72}$$

which satisfies (4.1.3). Thus, Eq. (4.3.72) defines the homomorphism $\overline{\mathcal{B}}_M \to \hat{\mathcal{B}}_M$.

This homomorphism of affine braid groups is readily carried over to the Hecke algebra case. Indeed, the definition (4.3.61) of the additional generator $\sigma_M$ (needed to close the set of the generators $\sigma_k \in H_M(q)$ to the periodic chain) looks like the similarity transformation of $\sigma_1$. Thus, the characteristic Hecke identity (4.3.1) for the elements $\sigma_1$ and $\sigma_M$ coincides.

*4.3.6. q-Dimensions of idempotents in $H_M(q)$ and knot/link polynomials*

Here we follow the approach presented in [208, 209]. Consider a linear map $\mathrm{Tr}_{D(n+1)}$: $H_{n+1}(q) \to H_n(q)$ from the Hecke algebra $H_{n+1}(q)$ to its subalgebra $H_n(q)$ which is defined by formulas (4.3.69), where we take $y_1 = 1$ (it means that $Z^{(k)} = Z^{(0)}$, $\forall k$) and fix the constant

$$Z^{(0)} = \frac{1-q^{-2d}}{q-q^{-1}}, \qquad Z^{(0)} \equiv \mathrm{Tr}_{D(n)}(\mathbf{1}), \tag{4.3.73}$$

for later convenience. Then one can define an Ocneanu's trace $\mathcal{T}r^{(M)}$: $H_M(q) \to \mathbb{C}$ as a sequence of maps

$$\mathcal{T}r^{(M)} := \mathrm{Tr}_{D(1)}\mathrm{Tr}_{D(2)} \cdots \mathrm{Tr}_{D(M)}. \tag{4.3.74}$$

**Proposition 4.22.** *The Jucys–Murphy elements $y_k \in H_{M+1}$ satisfy the following identity [208, 209]:*

$$1 + \lambda \, \mathrm{Tr}_{D(M+1)} \left( \frac{y_{M+1}}{t - y_{M+1}} \right) = \frac{(t-q^{-2d})}{(t-1)} \prod_{k=1}^{M} \frac{(t-y_k)^2}{(t-q^2 y_k)(t-q^{-2}y_k)}, \tag{4.3.75}$$

*where $\lambda = q - q^{-1}$ and $t$ is a parameter.*

**Proof.** Taking into account the definition (4.3.8) of the generators $y_M$, we have the equations

$$\frac{1}{(t-y_{M+1})}\sigma_M^{-1} = \sigma_M^{-1}\frac{1}{(t-y_M)} + \frac{\lambda \, y_M}{(t-y_M)}\frac{1}{(t-y_{M+1})}, \tag{4.3.76}$$

$$\frac{1}{(t-y_{M+1})}\sigma_M = \sigma_M^{-1}\frac{1}{(t-y_M)} + \frac{\lambda \, t}{(t-y_M)}\frac{1}{(t-y_{M+1})}. \tag{4.3.77}$$

We multiply the both sides of Eq. (4.3.76) from the right by $\sigma_M$. Then, in the r.h.s. of the result, we substitute Eq. (4.3.77) and apply the map $\mathrm{Tr}_{D(M+1)}$ (4.3.69). Finally, we obtain a recurrence relation

$$\frac{(t-q^2 y_M)(t-q^{-2}y_M)}{(t-y_M)^2} Z_{M+1} = Z_M + \frac{\lambda \, y_M \, (1-\lambda \, Z^{(0)})}{(t-y_M)^2}, \qquad Z_M := \mathrm{Tr}_{D(M)}\Big(\frac{1}{t-y_M}\Big), \tag{4.3.78}$$

where the parameter $Z^{(0)}$ is introduced in (4.3.69), (4.3.73). Equation (4.3.78) is simplified by the substitution $Z_M = \tilde{Z}_M - (1-\lambda \, Z^{(0)})/(\lambda \, t)$ and we have

$$\frac{(t-q^2 y_M)(t-q^{-2}y_M)}{(t-y_M)^2} \, \tilde{Z}_{M+1} = \tilde{Z}_M, \qquad \tilde{Z}_1 = \frac{1}{\lambda \, t}\Big(1 + \frac{\lambda \, Z^{(0)}}{(t-1)}\Big).$$

This equation can be easily solved and finally we obtain the formula

$$Z_{M+1} = \frac{1}{\lambda \, t}\left(1 + \frac{\lambda \, Z^{(0)}}{(t-1)}\right) \prod_{k=1}^{M} \frac{(t-y_k)^2}{(t-q^2 y_k)(t-q^{-2}y_k)} - \frac{1}{\lambda \, t}\left[1 - \lambda \, Z^{(0)}\right],$$

which is equivalent to (4.3.75). ∎

We note that the r.h.s. of (4.3.75) is the symmetric function in $y_k$ ($k = 1, \ldots, M$). It means that the element (4.3.75) belongs to the center of the Hecke algebra $H_M \subset H_{M+1}$.

**Proposition 4.23.** *Ocneanu's traces of idempotents $e(T_\Lambda)$ and $e(T'_\Lambda)$, corresponding to different Young tableaux $T_\Lambda$ and $T'_\Lambda$ of the same shape $\Lambda \vdash M$, coincide. Thus, characteristics*

$$\mathrm{qdim}(\Lambda) := \mathcal{T}r^{(M)} e(T_\Lambda) = \mathcal{T}r^{(M)} e(T'_\Lambda), \tag{4.3.79}$$

*depending only on the Young diagram $\Lambda$, are called q-dimension of $\Lambda \vdash M$ and we have [203]*

$$\mathrm{qdim}(\Lambda) = q^{-M\,d} \prod_{n,m\in\Lambda} \frac{[d+m-n]_q}{[h_{n,m}]_q}, \qquad [h]_q := \frac{q^h - q^{-h}}{q - q^{-1}}. \tag{4.3.80}$$

*Here $h_{n,m}$ are hook lengths of nodes $(n,m)$ of the diagrams $\Lambda$, the product runs over all nodes of $\Lambda$ and the constant $d$ is defined in (4.3.73).*

**Proof.** We follow the proof presented in [208, 209]. Idempotents $e(T_\Lambda)$ and $e(T'_\Lambda)$, corresponding to two different tableaux $T_\Lambda$ and $T'_\Lambda$ having the same shape $\Lambda$, are related by several similarity transformations with operators $U_j$ (see the l.h.s of (4.3.48)). This implies (4.3.79).

To calculate the characteristic "qdim" (4.3.79) for the diagram (4.3.34) with $M$ nodes and $n$ rows, we find the right action to the both sides of (4.3.75) by the idempotent $e(T_\Lambda)$, where $T_\Lambda$ is any Young tableau of the shape of Young diagram (4.3.34). We take the "row-standard" tableau $T_\Lambda$ corresponding to the eigenvalues of $y_k$ arranged along the rows from left to right and from top to bottom:

$$\begin{array}{l} y_1 = 1,\ y_2 = q^2,\ y_3 = q^4,\ \dots,\ y_{\lambda_1 - 1} = q^{2(\lambda_1-2)},\ y_{\lambda_1} = q^{2(\lambda_1-1)}, \\ y_{\lambda_1+1} = q^{-2},\ y_{\lambda_1+1} = 1,\ \dots,\ y_{\lambda_1+\lambda_2} = q^{2(\lambda_2-2)}, \\ \dots\dots\dots\dots\dots\dots, \\ y_{M-\lambda_n+1} = q^{-2(n-1)},\ \dots,\ y_M = q^{2(\lambda_n - n)}, \end{array} \tag{4.3.81}$$

where $n$ is the number of rows in $\Lambda$. After substitution of the eigenvalues (4.3.81) into the r.h.s. of (4.3.75), which is the product over all $M$ nods of the Young diagram (4.3.34), and cancelation of many factors, we obtain the result ($n_k = n,\ \ n_0 := 0$):

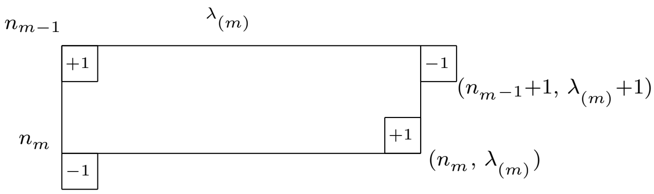
$$\mathrm{Tr}_{D(M+1)} \left( \sum_j P_j \frac{(q-q^{-1})\,\mu_j}{t - \mu_j} \right) = e(T_\Lambda) \left( \frac{t - q^{-2d}}{t - q^{-2n}} \prod_{r=1}^{k} \frac{t - q^{2(\lambda_{(r)} - n_r)}}{t - q^{2(\lambda_{(r)} - n_{r-1})}} - 1 \right). \tag{4.3.82}$$

We inserted into the l.h.s. of (4.3.75) the spectral decomposition of the idempotent $e(T_\Lambda)$ (see (4.3.35)):

$$e(T_\Lambda) = e(T_\Lambda) \sum_j \Pi_j = \sum_j P_j, \qquad P_j\, y_{M+1} = P_j\, \mu_j, \qquad \mu_j := q^{2(\lambda_{(j)} - n_{j-1})}.$$

The idempotent $P_j = e(T_{\Lambda^{(j)}}) \in H_{M+1}$ projects $y_{M+1}$ on its eigenvalue $\mu_j$ which also appeared in the denominator in the r.h.s. of (4.3.82) for $r = j$.

Let us discuss in more detail how one can deduce the expression in the r.h.s. of (4.3.82). It is obtained if we evaluate the action of the idempotent $e(T_\lambda)$ on the element in the r.h.s. of (4.3.75) for each rectangular block in the diagram $\Lambda$ (4.3.34) with all rows having the same length $\lambda_{(m)}$ and the number of rows equal to $(n_m - n_{m-1})$. The result of such evaluation, given in the r.h.s. of (4.3.82), is the product of the factor $\frac{t-q^{-2d}}{t-q^{-2n}}$ and all factors which are visualized as figure in (4.3.83) and correspond to all rectangular blocks in the diagram (4.3.34)):




$$= \prod_{i=1}^{4} (t-\mu'_i)^{\alpha_i} = \frac{(t-q^{-2n_{m-1}})(t-q^{2(\lambda_{(m)}-n_m)})}{(t-q^{-2n_m})(t-q^{2(\lambda_{(m)}-n_{m-1})})} \qquad (4.3.83)$$

Each rectangular block contributes to the r.h.s. of (4.3.82) four factors which correspond to four cells indicated in (4.3.83) by indices $\alpha_i = \pm 1$ and having contents $\mu'_i$ ($i = 1, \dots, 4$). Indices $\alpha_i = \pm 1$ are powers of the factors $(t-\mu'_i)^{\alpha_i}$, in the r.h.s. of (4.3.83). Two factors which correspond to sells with contents $(-n_{m-1})$ and $(-n_m)$ are canceled in the r.h.s. of (4.3.82) by neighboring blocks from top and bottom if any. The other cells of the block (4.3.83) have the powers equal to zero, the corresponding factors are canceled and do not contribute to the r.h.s. of (4.3.82).

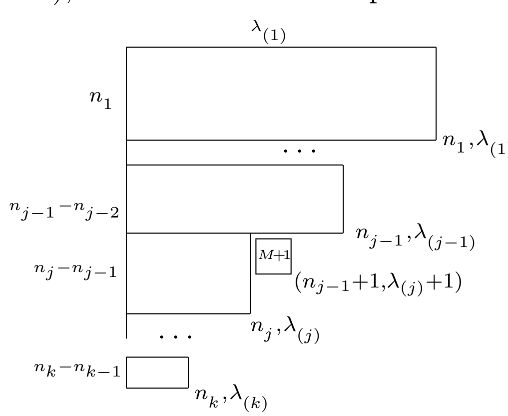
Now we compare the residues at $t = \mu_j$ in both sides of Eq. (4.3.82) and deduce

$$\mathrm{Tr}_{D(M+1)}\big(e(T_{\Lambda^{(j)}})\big) = \frac{\mu_j^{-1}}{(q-q^{-1})} e(T_\Lambda) \left. \frac{(t-q^{-2d}) \prod\limits_{r=1}^{k} (t-q^{2(\lambda_{(r)}-n_r)})}{(t-q^{-2n}) \prod\limits_{\substack{r\neq j \\ r=1}}^{k} (t-q^{2(\lambda_{(r)}-n_{r-1})})} \right|_{t=\mu_j\equiv q^{2(\lambda_{(j)}-n_{j-1})}} =$$

$$= e(T_\Lambda)\cdot q^{-d}\,[q^{(\lambda_{(j)}-n_{j-1}+d)}]_q \, \frac{\prod_{n,m\in\Lambda} [h_{n,m}]_q}{\prod_{n,m\in\Lambda^{(j)}} [h'_{n,m}]_q}, \qquad (4.3.84)$$

where $h_{n,m}$ and $h'_{n,m}$ are hook lengths[25] of nodes $(n, m)$ of the diagrams $\Lambda \vdash M$ and $\Lambda^{(j)} \vdash (M+1)$. The diagram $\Lambda^{(j)}$ is obtained by adding to $\Lambda$ (shown in (4.3.34)) a new node with coordinates $(n_{j-1}+1, \lambda_{(j)}+1)$, as it is shown in the picture:

$\Lambda^{(j)} =$




(4.3.85)

To deduce the last formula in (4.3.84), we need to check the identity

$$q^{(n_{j-1}-\lambda_{(j)})} \frac{(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(j)}-n_j)})}{(q^{2(\lambda_{(j)}-n_{j-1})} - q^{-2n_k})} \prod_{\substack{r=1 \\ r\neq j}}^{k} \frac{(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(r)}-n_r)})}{(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(r)}-n_{r-1})})} = \frac{\prod\limits_{(r,m)\in\Lambda} [h_{r,m}]_q}{\prod\limits_{(r,m)\in\Lambda^{(j)}} [h'_{r,m}]_q}, \qquad (4.3.86)$$

where $n_0 \equiv 0$, $n_k \equiv n$, while $h_{r,m}$ and $h'_{r,m}$ are hook lengths for cells with coordinates $(r, m)$ in the diagrams $\Lambda$ and $\Lambda^{(j)}$, respectively. The products in the r.h.s. of (4.3.86) run over all cells

[25] The hook length of the node $(n, m)$ of the diagram $\Lambda = [\lambda_1, \lambda_2, \dots]$ is defined as $h_{n,m} = (\lambda_n + \lambda^\vee_m - n - m + 1)$, where $\Lambda^\vee = [\lambda^\vee_1, \lambda^\vee_2, \dots]$ is the transpose partition of the partition $\Lambda$.

of $\Lambda$ and $\Lambda^{(j)}$. To prove (4.3.86), we note that the lengths of hooks $h_{r,m}$ and $h'_{r,m}$ for diagrams $\Lambda$ and $\Lambda^{(j)}$ differ only for cells, for which $r = n_{j-1}+1$, or $m = \lambda_{(j)}+1$, i.e., for cells located in the same row, or in the same column with additional cell $(n_{j-1}+1, \lambda_{(j)}+1)$. Thus, we have

$$\frac{\prod\limits_{r,m\in\lambda_n} [h_{r,m}]_q}{\prod\limits_{r,m\in\lambda_{n+1}} [h'_{r,m}]_q} = \prod_{r=1}^{n_{j-1}} \frac{[h_{r,\lambda_{(j)}+1}]_q}{[h'_{r,\lambda_{(j)}+1}]_q} \prod_{m=1}^{\lambda_{(j)}} \frac{[h_{n_{j-1}+1,m}]_q}{[h'_{n_{j-1}+1,m}]_q}. \tag{4.3.87}$$

Further, if rows with numbers $r$ and $r+1$ in the diagram $\Lambda$ have the same length, then we obviously have $h_{r,\lambda_{(j)}+1} = h'_{r+1,\lambda_{(j)}+1}$. And analogously, if the columns with numbers $m$ and $m+1$ in the diagram $\Lambda$ have the same height, then we have $h_{n_{j-1}+1,m} = h'_{n_{j-1}+1,m+1}$. That is why a lot of factors in the r.h.s. of (4.3.87) are canceled, according to the block form of diagram (4.3.34), and we obtain contributions only from the first and the last row in the blocks (located above the additional cell in $\Lambda^{(j)}$) of the diagram $\Lambda$:

$$\prod_{r=1}^{n_{j-1}} \frac{[h_{r,\lambda_{(j)}+1}]_q}{[h'_{r,\lambda_{(j)}+1}]_q} = \prod_{p=1}^{j-1} \frac{[h_{n_p,\lambda_{(j)}+1}]_q}{[h'_{n_{p-1}+1,\lambda_{(j)}+1}]_q} = q^{n_{j-1}} \prod_{p=1}^{j-1} \frac{(q^{2(\lambda_{(p)}-n_p)} - q^{2(\lambda_{(j)}-n_{j-1})})}{(q^{2(\lambda_{(p)}-n_{p-1})} - q^{2(\lambda_{(j)}-n_{j-1})})}, \tag{4.3.88}$$

and contributions only from the first and the last column in the blocks (located to the left of the additional cell) of the diagram $\lambda_n$:

$$\prod_{m=1}^{\lambda_{(j)}} \frac{[h_{n_{j-1}+1,m}]_q}{[h'_{n_{j-1}+1,m}]_q} = \prod_{p=j}^{k} \frac{[h_{n_{j-1}+1,\lambda_{(p)}}]_q}{[h'_{n_{j-1}+1,\lambda_{(p+1)}+1}]_q} = q^{-\lambda_{(j)}} \prod_{p=j}^{k} \frac{(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(p)}-n_p)})}{(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(p+1)}-n_p)})}. \tag{4.3.89}$$

The substitution of (4.3.88) and (4.3.89) into (4.3.87) gives (4.3.86). Finally, we apply Ocneanu's trace $\mathcal{T}r^{(M)}$ to both sides of Eq. (4.3.84) and find the recurrence relation

$$\mathrm{qdim}(\Lambda^{(j)}) = \mathrm{qdim}(\Lambda)\; q^{-d}\; [\lambda_{(j)} - n_{j-1} + d]_q \; \frac{\prod\limits_{n,m\in\Lambda} [h_{n,m}]_q}{\prod\limits_{n,m\in\Lambda^{(j)}} [h'_{n,m}]_q},$$

which is uniquely solved (up to a constant multiplier[26]) as in (4.3.80). ■

The $R$-matrix representations (see [42, 113]) of the Hecke algebra $H_{M+1}(q)$ were discussed in Subsection **3.4** in the context of the quantum group $GL_q(N)$ and in Subsection **3.7** in the context of the quantum supergroup $GL_q(N|K)$. For the $R$-matrices (3.7.1) related to the quantum supergroup $GL_q(N|K)$, the parameter $d$ is equal to $(N-K)$. This fact follows from Eqs. (4.3.69) and (4.3.73) in the limit $q \to 1$. It also justifies our choice of the parametrization of $Z_0$ in (4.3.73).

The statement (4.3.79) in Proposition **4.23** can be generalized. Let $T$ be a quantum matrix satisfying

$$\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12}, \tag{4.3.90}$$

where $\hat{R}_{12} = \rho(\sigma_1)$ is the $R$-matrix representation of the Hecke algebra.

[26] We fix this multiplier by the condition $\mathrm{qdim}(\square) = q^{-d}[d]_q = Z^{(0)}$; see (4.3.73).

**Proposition 4.24.** *The quantum traces (for the definition of the quantum traces see Subsection* **3.1.2***) of the matrices* $[T_1 \cdots T_{_M}\, \rho(e(T_\Lambda))]$ *and* $[T_1 \cdots T_{_M}\, \rho(e(T'_\Lambda))]$*, where different tableaux* $T_\Lambda$ *and* $T'_\Lambda$ *are of the same shape* $\Lambda \vdash M$*, coincide:*

$$\chi_\Lambda(T) := \mathrm{Tr}_{\mathcal{D}(1\dots M)} \left(T_1 \cdots T_M\, \rho(e(T_\Lambda))\right) = \mathrm{Tr}_{\mathcal{D}(1\dots M)} \left(T_1 \cdots T_M\, \rho(e(T'_\Lambda))\right). \tag{4.3.91}$$

*Thus, the element* $\chi_\Lambda(T)$ *depends only on the shape of the diagram* $\Lambda$*.*

According to Proposition **3.8** (see Subsection **3.2.4**), the elements $\chi_\Lambda(T)$ for all Young diagrams $\Lambda \vdash M$ $(M = 1, 2, 3, \dots)$ generate the commutative subalgebra in the $RTT$ algebra (4.3.90).

Consider the $GL_q(N)$ quantum group (4.3.90) with the standard $GL_q(N)$ Drinfeld–Jimbo $\hat{R}_{12}$-matrix (3.4.9). As we mentioned above (see Subsection **3.4** and [42, 113]), this standard $GL_q(N)$ matrix $\hat{R}_{12}$ ($R$-matrix in the defining representation) gives the representation of the Hecke algebra. We note that the $GL_q(N)$ quantum matrix $T$ can be realized by arbitrary numerical diagonal $(N \times N)$ matrix $Y = \mathrm{diag}(x_1, \dots, x_N)$. Then $\chi_\Lambda(Y)$ is a numerical function of the deformation parameter $q$ and the entries $\{x_i\}$ of $Y$. In the classical limit $q \to 1$, the operator $\rho(e(T_\Lambda))$ tends to the Young projector, and the function $\chi_\Lambda(Y)$ coincides with a character of the element $Y \in GL(N)$ in the representation corresponding to the diagram $\Lambda$; i.e., $\chi_\Lambda(Y)|_{q\to 1}$ coincides with the Schur polynomial $S_\Lambda(x_1, \dots, x_N)$.

**Remark 1.** The hook formula (4.3.80) for the $q$-dimension of $\Lambda \vdash M$ is written in the remarkable form (which is more convenient for calculations):

$$\mathrm{qdim}(\Lambda) = q^{-M\,d} \prod_{i=1}^{k} \frac{[d+i-1]_q!}{[d-\lambda_i^\vee + i - 1]_q! [\lambda_i^\vee + k - i]_q!} \prod_{i<j} [\lambda_i^\vee - \lambda_j^\vee + (j-i)]_q, \tag{4.3.92}$$

where $\Lambda^\vee = (\lambda_1^\vee, \lambda_2^\vee, \dots, \lambda_k^\vee)$ is the transpose partition of $\Lambda$ and $[h]_q := \frac{q^h - q^{-h}}{q - q^{-1}}$.

**Remark 2.** At the end of this subsection, we derive a universal analogue (in terms of Hecke algebra generators) of the formula (3.1.57) for knot/link invariants. Let $B_{1\to M}$ be monomial written as a product of generators $\sigma_i \in H_M$. It is clear that $B_{1\to M}$ is visualized as a braid with $M$ strands. Then, by means of the Ocneanu's trace (4.3.74), we construct the Hecke algebraic analog of (3.1.57) in the form

$$\mathsf{Q}(B_{1\to M}) := \mathcal{T}r^{(M)}\big(B_{1\to M}\big). \tag{4.3.93}$$

Insert in the right-hand side of (4.3.93) the resolution of the unit operator (see the second equation in (4.3.2))

$$1 = \sum_{\Lambda \vdash M} \sum_{\mathsf{T}_\Lambda^{\vec{a}}} e(\mathsf{T}_\Lambda^{\vec{a}}) = \sum_{\vec{a}} P(X_{\vec{a}}), \tag{4.3.94}$$

where $e(\mathsf{T}_\Lambda^{\vec{a}}) = P(X_{\vec{a}})$ are mutually orthogonal idempotents related to the standard Young tableau $\mathsf{T}_\Lambda^{\vec{a}}$ (with a content $\vec{a} = (a_1, \dots, a_M)$) having the shape of the Young diagram $\Lambda \vdash M$ (or related to the path $X_{\vec{a}}$ in the coloured Young graph for $H_M$). The sum in the r.h.s. of (4.3.94) is going over all standard tableaux with $M$ nodes, or equivalently over all their contents $\vec{a} \in \mathrm{Spec}(y_1, \dots, y_M)$. As a result, we obtain for knot/link invariants (4.3.93) the expressions

$$\begin{aligned} \mathsf{Q}(B_{1\to M}) &= \sum_{\vec{a}} \mathcal{T}r^{(M)}\big(B_{1\to M}\, P(X_{\vec{a}})\big) = \sum_{\vec{a}} \mathcal{T}r^{(M)}\big(P(X_{\vec{a}})\, B_{1\to M}\, P(X_{\vec{a}})\big) = \\ &= \sum_{\vec{a}} C_{\vec{a}}(B_{1\to M}) \mathcal{T}r^{(M)}\big(P(X_{\vec{a}})\big) = \sum_{\Lambda \vdash \mathsf{M}} \mathrm{qdim}(\Lambda) \sum_{\vec{a}(\Lambda)} C_{\vec{a}(\Lambda)}(B_{1\to M}), \end{aligned} \tag{4.3.95}$$

where we used relation $P(X_{\vec{a}})^2 = P(X_{\vec{a}})$, cyclic property of $\mathcal{T}r^{(M)}$, the identity (4.3.53) for the diagonal matrix units of $H_M$ and substitute $\mathcal{T}r^{(M)}\big(P(X_{\vec{a}(\Lambda)})\big) = \mathrm{qdim}(\Lambda)$ for the Young diagrams $\Lambda \vdash M$. In the last equality of (4.3.95), we split the sum over all contents $\vec{a}$ of the standard Young tableaux with $M$ nodes into the sum over Young diagrams $\Lambda \vdash M$ and the sum over all contents $\vec{a}(\Lambda)$ of the Young tableaux $\mathsf{T}_\Lambda$ having the fixed shape $\Lambda \vdash M$. We note that in (4.3.95) the q-dimensions $\mathrm{qdim}(\Lambda)$ (given by formula (4.3.80)) are independent of the braid $B_{1\to M}$ and all dependence on $B_{1\to M}$ is contained in the coefficients $C_{\vec{a}(\Lambda)}(B_{1\to M})$. As was indicated in Subsection **4.3.4**, the coefficients $C_{\vec{a}(\Lambda)}(B_{1\to M})$ can be explicitly calculated with the help of Eqs. (4.3.55), (4.3.56), and (4.3.58). The $R$-matrix version [221] of the formula (4.3.95) is extensively used in [222–227] (see also references therein) for calculations of HOMFLY ($GL_q(N)$, $N = d$) knot/link polynomials.

### *4.4. Birman–Murakami–Wenzl algebras $BMW_{M+1}(q,\nu)$*

#### *4.4.1. Definition*

The Birman–Murakami–Wenzl algebra $BMW_{M+1}(q,\nu)$ is generated by the elements $\kappa_i|_{i=1,\dots,M}$ and invertible elements $\sigma_i|_{i=1,\dots,M}$ which satisfy (4.1.1) and the following relations [228, 229, 244]:

$$\kappa_i\,\sigma_i = \sigma_i\,\kappa_i = \nu\,\kappa_i, \tag{4.4.1}$$

$$\kappa_i\,\sigma_{i-1}^{\pm1}\,\kappa_i = \nu^{\mp1}\,\kappa_i, \tag{4.4.2}$$

$$\sigma_i - \sigma_i^{-1} = \lambda\,(1-\kappa_i), \tag{4.4.3}$$

where $\nu \in \mathbb{C}\backslash\{0, \pm q^{\pm1}\}$ is an additional parameter of the algebra; $\lambda = q - q^{-1}$. The following relations can be derived from (4.1.1), (4.4.1)–(4.4.3):

$$\kappa_i\,\kappa_i = \mu\,\kappa_i, \tag{4.4.4}$$

$$(\;\mu = (\lambda + \nu^{-1} - \nu)/\lambda = -(\nu + q^{-1})(\nu - q)\,(\lambda\nu)^{-1}\;), \tag{4.4.5}$$

$$\kappa_i\,\sigma_{i\pm1}\,\sigma_i = \sigma_{i\pm1}\,\sigma_i\,\kappa_{i\pm1}, \tag{4.4.6}$$

$$\kappa_i\,\sigma_{i\pm1}\,\sigma_i = \kappa_i\,\kappa_{i\pm1}, \tag{4.4.7}$$

$$\kappa_i\,\sigma_{i\pm1}^{-1}\,\sigma_i^{-1} = \kappa_i\,\kappa_{i\pm1}, \tag{4.4.8}$$

$$\sigma_{i\pm1}\,\kappa_i\,\sigma_{i\pm1} = \sigma_i^{-1}\,\kappa_{i\pm1}\,\sigma_i^{-1}, \tag{4.4.9}$$

$$\kappa_i\,\kappa_{i\pm1}\,\kappa_i = \kappa_i, \tag{4.4.10}$$

$$\kappa_{i\pm1}\,\kappa_i\,(\sigma_{i\pm1} - \lambda) = \kappa_{i\pm1}\,(\sigma_i - \lambda), \tag{4.4.11}$$

$$(\sigma_i - \lambda)\,\kappa_{i-1}\,(\sigma_i - \lambda) = (\sigma_{i-1} - \lambda)\,\kappa_i\,(\sigma_{i-1} - \lambda). \tag{4.4.12}$$

Equation (4.4.4) is deduced by the action of the element $\kappa_i$ on (4.4.3) and using (4.4.1). Relations (4.4.6) follow from (4.1.1) and (4.4.3). Relations (4.4.7) and (4.4.8) with lower signs are obtained by multiplying (4.4.2) with $\sigma_{i-1}^{\mp1}\sigma_i^{\mp1}$ from the right and using (4.4.1) and (4.4.6). Equation (4.4.9) follows from (4.4.7), (4.4.8). Combining the pair of relations (4.4.2) in the form: $\kappa_i\,(\sigma_{i+1} - \sigma_{i+1}^{-1})\,\kappa_i = (\nu^{-1} - \nu)\,\kappa_i$ and using (4.4.3) and (4.4.4), we derive (4.4.10). Equation (4.4.11) is proved as follows:

$$\kappa_{i\pm1}\,\kappa_i\,(\sigma_{i\pm1} - \lambda) = \kappa_{i\pm1}\,\kappa_i\,(\sigma_{i\pm1}^{-1} - \lambda\kappa_{i\pm1}) = \kappa_{i\pm1}\,(\sigma_i - \lambda),$$

where we have used (4.4.3), (4.4.7), and (4.4.10). Equation (4.4.12) is deduced by means of Eq. (4.4.11), its mirror counterpart, and Eq. (4.4.10). The pairs of equations in (4.4.6)–(4.4.10) (with upper and lower signs) are related to each other by the similarity transformations

$$\sigma_{i+1} = V_i\sigma_{i-1}V_i^{-1}, \quad \sigma_i = V_i\sigma_iV_i^{-1},$$

where $V_i = \sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i-1}\sigma_i$ (the only braid relations (4.1.1) should be used). We also present the relations

$$\kappa_{i\pm1}\,\kappa_i\,(\sigma_{i\pm1}^{-1}+\lambda) = \kappa_{i\pm1}\,(\sigma_i^{-1}+\lambda), \tag{4.4.13}$$

$$(\sigma_i^{-1}+\lambda)\,\kappa_{i-1}\,(\sigma_i^{-1}+\lambda) = (\sigma_{i-1}^{-1}+\lambda)\,\kappa_i\,(\sigma_{i-1}^{-1}+\lambda), \tag{4.4.14}$$

which are related to (4.4.11), (4.4.12) via the obvious isomorphism $(\sigma_i, q, \nu) \leftrightarrow (\sigma_i^{-1}, q^{-1}, \nu^{-1})$ of the algebras $BMW_{M+1}(q,\nu) \simeq BMW_{M+1}(q^{-1},\nu^{-1})$. This isomorphism can be checked by the substitution $\sigma_i \to \sigma_i^{-1}$ in (4.1.1), (4.4.1)–(4.4.3).

In fact, the pair of relations (4.4.2) (in the definition of the Birman–Murakami–Wenzl algebra) is not independent for the case $\nu \neq \lambda$ [136]. Indeed, using $\kappa_i\,\sigma_{i-1}\,\kappa_i = \nu^{-1}\,\kappa_i$ and (4.1.1), one can deduce $\sigma_{i-1}\sigma_i\kappa_{i-1}\kappa_i = \nu\kappa_i\sigma_{i-1}\kappa_i = \kappa_i$, which is written in the form $\sigma_{i-1}^{-1}\kappa_i = \sigma_i\kappa_{i-1}\kappa_i$. Acting to this relation by $\lambda\kappa_i$ from the left, we deduce

$$\lambda\kappa_i\sigma_{i-1}^{-1}\kappa_i = \lambda\nu\kappa_i\kappa_{i-1}\kappa_i = \nu\kappa_i(\sigma_{i-1}^{-1} - \sigma_{i-1} + \lambda)\kappa_i = \nu\kappa_i\sigma_{i-1}^{-1}\kappa_i + \nu(\lambda\mu - \nu^{-1})\kappa_i,$$

which is equivalent to $(\lambda-\nu)(\kappa_i\sigma_{i-1}^{-1}\kappa_i - \nu\kappa_i) = 0$ and, thus, to the above statement.

The $BMW_M(q,\nu)$ algebras are $q$-deformations of the Brauer algebras $\mathcal{B}r_M(\omega)$ (for the definition of the Brauer algebras see, e.g., [102, 139] and references therein) and $\dim BMW_M(q,\nu) = (2M-1)!!$ (for general parameters $q, \nu$).

#### *4.4.2. Symmetrizers, antisymmetrizers and Baxterized elements in $BMW_{M+1}$*

Below, for brevity, we often omit in the notation $BMW_{M+1}(q,\nu)$ the dependence on the parameters $q,\nu$. One can construct the analogs of the symmetrizers and antisymmetrizers for the algebra $BMW_{M+1}$ using the inductive relations similar to that we have considered in the Hecke case (4.3.43):

$$S_{1\to n} = f^{(-)}_{1\to n}\, S_{1\to n-1} = S_{1\to n-1}\, \overline{f}^{(-)}_{1\to n}, \tag{4.4.15}$$

$$A_{1\to n} = f^{(+)}_{1\to n}\, A_{1\to n-1} = A_{1\to n-1}\, \overline{f}^{(+)}_{1\to n}, \tag{4.4.16}$$

where 1-shuffles are

$$f^{(\pm)}_{1\to n} = \frac{1}{[n]_q!}\,\sigma_1^{(\pm)}(q^{\pm1})\cdots\sigma_{n-2}^{(\pm)}(q^{\pm(n-2)})\,\sigma_{n-1}^{(\pm)}(q^{\pm(n-1)}),$$

$$\overline{f}^{(\pm)}_{1\to n} = \frac{1}{[n]_q!}\,\sigma_{n-1}^{(\pm)}(q^{\pm(n-1)})\,\sigma_{n-2}^{(\pm)}(q^{\pm(n-2)})\cdots\sigma_1^{(\pm)}(q^{\pm1}),$$

and $\sigma_i^{(\pm)}(x)$ are Baxterized elements (cf. (3.12.14), (3.12.15), (3.12.16)) for the algebra $BMW_{M+1}(q,\nu)$ (see [200, 237–239], [46]):

$$\sigma_i^{(\pm)}(x) = \frac{1}{\lambda}(x^{-1}\,\sigma_i - x\,\sigma_i^{-1}) + \frac{(\nu \pm q^{\pm1})}{(\nu x \pm q^{\pm1}x^{-1})}\,\kappa_i = \tag{4.4.17}$$

$$= \frac{x^{-1}-x}{\lambda}\left(\sigma_i + \frac{\lambda}{(x^{-2}-1)}\,\mathbf{1} + \frac{\nu\lambda}{(\nu - a x^{-2})}\,\kappa_i\right)\Big|_{a=\mp q^{\pm1}} = \tag{4.4.18}$$

$$= x\left(\mathbf{1} + \frac{1}{\lambda}(x^{-2}-1)\,\sigma_i\right)\left(1 + \frac{\lambda(x^{-2}-1)}{(\lambda - \nu(1-x^{-2}))(1 \pm q^{\pm1}\nu^{-1}x^{-2})}\,\kappa_i\right).$$

These elements are normalized by the conditions $\sigma^{(\pm)}(\pm 1) = \pm\mathbf{1}$, satisfy the Yang–Baxter equations

$$\sigma_n^{(\pm)}(x)\,\sigma_{n-1}^{(\pm)}(xy)\,\sigma_n^{(\pm)}(y) = \sigma_{n-1}^{(\pm)}(y)\,\sigma_n^{(\pm)}(xy)\,\sigma_{n-1}^{(\pm)}(x) \tag{4.4.19}$$

and obey

$$\sigma_i^{(\pm)}(x)\,\sigma_i^{(\pm)}(x^{-1}) = \left(1 - \lambda^{-2}\,(x - x^{-1})^2\right)\,\mathbf{1}. \tag{4.4.20}$$

Let parameter $a$ be $(-q)$, or $q^{-1}$ (see (4.4.18)), and we respectively denote $\sigma_i^{(a)}(x) = \sigma_i^{(\pm)}(x)$. Then the Baxterized elements (4.4.18) are written after an additional normalization in the form (cf. (4.3.42))

$$\sigma_i^{(a)\prime}(x) = \frac{\lambda x^2}{(a^{-1}x - ax^{-1})}\sigma_i^{(a)}(x) = \frac{(\sigma_i - a\,x^2)}{(\sigma_i - a\,x^{-2})}. \tag{4.4.21}$$

New normalized elements (4.4.21) obviously satisfy "unitarity conditions": $\sigma_i^{(a)\prime}(x)\;\sigma_i^{(a)\prime}(x^{-1}) = \mathbf{1}$ and $\sigma^{(a)\prime}(\pm 1) = \mathbf{1}$. Identities (4.4.17)–(4.4.21) are checked with the help of relations (4.4.1)–(4.4.12).

Note that the elements $\sigma_i^{(+)}(x)$ and $\sigma_i^{(-)}(x)$ (4.4.17) are related to each other by the transformation $q \leftrightarrow -q^{-1}$, which corresponds to the isomorphism of algebras $BMW_{M+1}(q,\nu) \simeq BMW_{M+1}(-q^{-1},\nu)$ and we also have

$$\sigma_i^{(+)}(x) - \sigma_i^{(-)}(x) = \frac{\nu(q+q^{-1})(x-x^{-1})}{(x\nu + qx^{-1})(x\nu - q^{-1}x^{-1})}\,\kappa_i.$$

We also stress that the both inequivalent sets $(\pm)$ of the Baxterized elements (4.4.17) are important for explicit constructions of (anti)symmetrizers (4.4.15), (4.4.16). To our knowledge, these both sets (4.4.17) were firstly presented in paper [237] (see also the very first version [46] of these lectures). The only one of these sets was presented in [200, 238] and in [239].

It follows from Eqs. (4.4.1)–(4.4.3) that the algebra $BMW_{M+1}(q,\nu)$ $(\nu \neq \lambda)$ is a quotient of the braid group algebra (4.1.1) if the additional relations on $\sigma_i$ are imposed:

$$(\sigma_i - q)(\sigma_i + q^{-1})(\sigma_i - \nu) = 0, \tag{4.4.22}$$

$$(\sigma_i^{-1} + \lambda - \sigma_i)\left(\sigma_{i+1}^{\pm 1}\,(\sigma_i^{-1} + \lambda - \sigma_i) - \lambda\nu^{\mp 1}\right) = 0.$$

This quotient is finite-dimensional and the dimension of $BMW_{M+1}(q,\nu)$ is $(2M+1)!! = 1 \cdot 3 \cdots (2M+1)$ (this dimension evidently follows from the graphical representation (3.10.36) of the $BMW_{M+1}(q,\nu)$ elements). The whole set of basis elements for the algebra $BMW_{M+1}(q,\nu)$ appears in the expansion of the symmetrizer $S_{M+1}$ (4.4.15) (or antisymmetrizer $A_{M+1}$ (4.4.16)). Note that the quotient of the Birman–Murakami–Wenzl algebra $BMW_{M+1}(q,\nu)$ (4.4.1)–(4.4.3) by an ideal generated by $\kappa_i$ is isomorphic to the $A$-type Hecke algebra $H_{M+1}(q)$.

The first symmetrizer and antisymmetrizer for the algebra $BMW_{M+1}(q,\nu)$ are (cf. Eqs. (3.10.5), (3.12.17)):

$$\begin{gathered} S_{1\to 2} = \tfrac{1}{[2]_q}\sigma_1^{(-)}(q^{-1}) = \tfrac{(\sigma_1^2 - q^{-2})(\sigma_1^2 - \nu^2)}{(q^2 - q^{-2})(q^2 - \nu^2)} = \\ = \tfrac{1}{[2]_q}(q^{-1} + \sigma_1 + \tfrac{\lambda}{1 - q\nu^{-1}}\kappa_1) = \tfrac{1}{q^2 - q^{-2}}(\sigma_1^2 - q^{-2})(1 - \mu^{-1}\,\kappa_1), \end{gathered} \tag{4.4.23}$$

$$\begin{gathered} A_{1\to 2} = \tfrac{1}{[2]_q}\sigma_1^{(+)}(q) = \tfrac{(\sigma_1^2 - q^2)(\sigma_1^2 - \nu^2)}{(q^{-2} - q^2)(q^{-2} - \nu^2)} = \\ = \tfrac{1}{[2]_q}(q - \sigma_1 - \tfrac{\lambda}{1 + q^{-1}\nu^{-1}}\kappa_1) = \tfrac{1}{q^{-2} - q^2}(\sigma_1^2 - q^2)(1 - \mu^{-1}\,\kappa_1). \end{gathered} \tag{4.4.24}$$

They are obviously orthogonal to each other and to the element $\kappa_1$ in view of the characteristic equation (4.4.22). The following equations also hold:

$$\sigma_1^{(-)}(q)\, S_{1\to 2} = 0 = \kappa_1\, S_{1\to 2}, \quad \sigma_1^{(+)}(q^{-1})\, A_{1\to 2} = 0 = \kappa_1\, A_{1\to 2},$$

which can be deduced from the "unitarity conditions" (4.4.20) and first equalities in (4.4.23), (4.4.24). In fact, these equations are special cases of the more general relations (for $i = 1,\dots,n-1$):

$$\sigma_i^{(-)}(q)\, S_{1\to n} = S_{1\to n}\, \sigma_i^{(-)}(q) = 0,$$

$$\sigma_i^{(+)}(q^{-1})\, A_{1\to n} = A_{1\to n}\, \sigma_i^{(+)}(q^{-1}) = 0,$$

which equivalent to the equations ($i = 1,\dots n-1$):

$$\begin{aligned} &(\sigma_i - q)\, S_{1\to n} = 0 = S_{1\to n}\, (\sigma_i - q), \quad \kappa_i\, S_{1\to n} = 0 = S_{1\to n}\, \kappa_i, \\ &(\sigma_i + q^{-1})\, A_{1\to n} = 0 = A_{1\to n}\, (\sigma_i + q^{-1}), \quad \kappa_i\, A_{1\to n} = 0 = A_{1\to n}\, \kappa_i, \end{aligned} \tag{4.4.25}$$

and demonstrate that $S_{1\to M+1}$, $A_{1\to M+1}$ are central idempotents. Equations (4.4.25) can be readily proved by means of the analogs of the factorization relations (4.2.6), (4.2.11) or by the induction using (4.4.15), (4.4.16) and the Yang–Baxter equations (4.4.19).

We note that the idempotents (4.4.15), (4.4.16) can be easily written in the form (cf. (4.3.36), (4.3.37))

$$S_{1\to n} = S_{1\to n-1}\, \tfrac{\sigma_{n-1}^{(-)}(q^{-(n-1)})}{[n]_q}\, S_{1\to n-1}, \tag{4.4.26}$$

$$A_{1\to n} = A_{1\to n-1}\, \tfrac{\sigma_{n-1}^{(+)}(q^{n-1})}{[n]_q}\, A_{1\to n-1}. \tag{4.4.27}$$

This inductive definition of the idempotents (4.4.15), (4.4.16) was also used in [131] and in [241] (see Lemma 7.6). Note that, in view of the definitions (4.4.17) of Baxterized elements $\sigma_k^{(\pm)}(x)$, expressions (4.4.26) and (4.4.27) have singularities for $q^{2k} = 1$, $\nu = q^{2k-3}$ and $q^{2k} = 1$, $\nu = -q^{-2k+3}$ ($k = 2,\dots,n$), respectively. It means that the representation theory of the BMW algebras has to be modified for $q^{2k} = 1$ and $\nu = \pm q^{\pm 2k-3}$.

Using the representations (4.4.26), (4.4.27), we prove the analog of Proposition **4.18** about symmetrizers and antisymmetrizers for the case of the Birman–Murakami–Wenzl algebra.

**Proposition 4.25.** *The idempotents $S_{1\to n}$ and $A_{1\to n}$ ($n = 2,\dots M+1$) (4.4.26), (4.4.27) for the Birman–Murakami–Wenzl algebra are expressed in terms of the Jucys–Murphy elements $y_k$ ($k = 2,\dots,M$):*

$$y_1 = 1, \quad y_{k+1} = \sigma_k y_k \sigma_k, \quad [y_k,\, y_m] = 0, \tag{4.4.28}$$

*as follows:*

$$S_{1\to n} = \prod_{i=2}^{n} \left( \frac{(y_i - q^{-2})}{(q^{2(i-1)} - q^{-2})}\, \frac{(y_i - \nu^2 q^{-2(i-2)})}{(q^{2(i-1)} - \nu^2 q^{-2(i-2)})} \right), \tag{4.4.29}$$

$$A_{1\to n} = \prod_{i=2}^{n} \left( \frac{(y_i - q^{2})}{(q^{-2(i-1)} - q^{2})}\, \frac{(y_i - \nu^2 q^{2(i-2)})}{(q^{-2(i-1)} - \nu^2 q^{2(i-2)})} \right). \tag{4.4.30}$$

**Proof.** To prove the identity (4.4.30), we show that it is equivalent to (4.4.27). The identity (4.4.29) for the symmetrizers (4.4.26) can be justified analogously. The equations (4.4.24) demonstrate that (4.4.30) coincide with (4.4.27) for $n = 2$. Then we use the induction. Let (4.4.30) coincide with the formula (4.4.27) for $A_{1\to n}$ for some fixed $n \geqslant 2$ and, thus, it is the element which satisfies (4.4.25). We prove that the formulas (4.4.27) and (4.4.30) are equivalent for $A_{1\to n+1}$. In view of the induction conjecture and obvious properties $[A_{1\to n},\, y_{n+1}] = 0$ (since $A_{1\to n}$ is a function of $y_i$) we obtain from (4.4.30):

$$A_{1\to n+1} = A_{1\to n}\, \frac{(y_{n+1} - q^2)}{(q^{-2n} - q^2)}\, \frac{(y_{n+1} - \nu^2 q^{2(n-1)})}{(q^{-2n} - \nu^2 q^{2(n-1)})}\, A_{1\to n}. \tag{4.4.31}$$

We need the identities

$$\begin{gathered}\sigma_n \dots \sigma_2 \sigma_1^2 \sigma_2 \dots \sigma_n\, \kappa_n = \nu^2\, \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1^{-2} \sigma_2^{-1} \dots \sigma_{n-1}^{-1}\, \kappa_n \;\Rightarrow \\ y_n\, \sigma_n\, y_n\, \kappa_n = \nu\, \kappa_n \quad \Rightarrow \quad y_{n+1}\, y_n\, \kappa_n = \nu^2\, \kappa_n,\end{gathered} \tag{4.4.32}$$

which follow from equation $\sigma_k \kappa_{k+1} = \sigma_{k+1}^{-1} \kappa_k \kappa_{k+1}$. We also deduce the analogs of the identities (4.3.8) for the Birman–Murakami–Wenzl algebra case:

$$\begin{gathered}y_{n+1} = \sigma_n \dots \sigma_2 \sigma_1^2 \sigma_2 \dots \sigma_n = 1 + \lambda \left( \sum_{i=1}^{n-1} \sigma_i \dots \sigma_{n-1} \sigma_n \sigma_{n-1} \dots \sigma_i + \sigma_n \right) - \\ -\lambda\nu \left( \sum_{i=1}^{n-1} \sigma_i^{-1} \dots \sigma_{n-1}^{-1} \kappa_n \sigma_{n-1}^{-1} \dots \sigma_i^{-1} + \kappa_n \right).\end{gathered} \tag{4.4.33}$$

Using Eqs. (4.4.32), (4.4.33) and $A_{1\to n} y_n = q^{2(1-n)} A_{1\to n}$ (see Eqs. (4.4.25) for $A_{1\to n}$), we obtain

$$A_{1\to n}\, y_{n+1}\, A_{1\to n} = A_{1\to n} \left( 1 + q\,(1 - q^{-2n})\, \sigma_n + \frac{\nu}{q}\,(1 - q^{2n}) \kappa_n \right) A_{1\to n}, \tag{4.4.34}$$

$$\begin{gathered}A_{1\to n}\, y_{n+1}^2\, A_{1\to n} = A_{1\to n} y_{n+1} \left( 1 + q\,(1 - q^{-2n})\, \sigma_n + \frac{\nu}{q}\,(1 - q^{2n}) \kappa_n \right) A_{1\to n} = \\ = A_{1\to n}[(1 + \lambda q(1 - q^{-2n})) + q(1 - q^{-2n})(q^2 + q^{-2n})\sigma_n + \\ + \tfrac{\nu}{q}\,(1 - q^{2n})(q^2 - q^{-2(n-1)} + q^{-2n} + \nu(\lambda + \nu q^{2(n-1)})\kappa_n)] A_{1\to n}.\end{gathered} \tag{4.4.35}$$

Then we substitute (4.4.34) and (4.4.35) into (4.4.31) and finally deduce

$$A_{1\to n+1} = \frac{q^{-1}\lambda}{(1 - q^{-2(n+1)})}\, A_{1\to n}\, \left( 1 + \frac{(q^{-2n} - 1)\sigma_n}{\lambda} + \frac{\nu(q^{-2n} - 1)\kappa_n}{(q^{-2n+1} + \nu)} \right) A_{1\to n}, \tag{4.4.36}$$

which coincides with (4.4.27). ■

One can prove directly the identities (4.4.25) for elements (4.4.29), (4.4.30). We again use the induction. Let (4.4.25) be valid for (4.4.30) for some fixed $n \geqslant 2$ (it is obviously correct for $n = 2$). Then we have to prove the identities (4.4.25) only for $n \to n+1$ and $i = n$. One can deduce

$$A_{1\to n}\, (y_{n+1} - \nu^2 q^{2(n-1)}) \kappa_n = A_{1\to n}\, (y_{n+1} - \nu^2\, y_n^{-1}) \kappa_n = 0, \tag{4.4.37}$$

where we have applied identities (4.4.32) and $A_{1\to n} y_n = A_{1\to n} q^{-2(n-1)}$. Using Eq. (4.4.37) and the relation $[A_{1\to n},\, y_{n+1}] = 0$, we prove that $A_{1\to n+1}\kappa_n = 0$ for (4.4.31). Now consider the following chain of relations:

$$\begin{gathered} A_{1\to n}\,(y_{n+1}-q^2)(\sigma_n+q^{-1}) = A_{1\to n}\,(\sigma_n y_n\sigma_n - q^2)(\sigma_n+q^{-1}) = \\ = A_{1\to n}\,(q\,\sigma_n y_n \sigma_n + \sigma_n y_n - \lambda\nu\sigma_n y_n\kappa_n - q^2\sigma_n - q) = \\ = A_{1\to n}\,(-\lambda\nu)\,\kappa_n\,\Bigg(\sum_{i=1}^{n-1}(-1)^{n-i}\,q^{i+1-n}\sigma_{n-1}^{-1}\dots\sigma_i^{-1}+q+ \\ +\sum_{i=1}^{n-2}(-1)^{n-i}\,q^{i-n}\kappa_{n-1}\sigma_{n-2}^{-1}\dots\sigma_i^{-1}+q\,\kappa_{n-1}+\nu q^{2(1-n)}\Bigg), \end{gathered} \tag{4.4.38}$$

where we have used Eqs. (4.4.25), (4.4.32), and (4.4.33). Multiplying Eq. (4.4.38) by the factor $(y_{n+1}-\nu^2 q^{2(n-1)})$ from the left and taking into account (4.4.37), we obtain $A_{1\to n+1}(\sigma_n+q^{-1}) = 0$.

**Remark 1.** The idempotents $S_{1\to n}$ and $A_{1\to n}$ for the Birman–Murakami–Wenzl algebra have been also constructed in another form in [240]. The authors of [240] (as well as the authors of [241]) have not used the Baxterized or Jucys–Murphy elements and, thus, their expressions for $S_{1\to n}$ and $A_{1\to n}$ look rather cumbersome. The construction of the primitive idempotents $S_{1\to n}$ and $A_{1\to n}$ in terms of the Baxterized elements (4.4.17) has been proposed by P. Pyatov in fall of 2001 and used, e.g., in [131]. After the substitution of (4.4.17) to (4.4.15), (4.4.16) and direct calculations one can derive the formulas for $S_{1\to n}$ and $A_{1\to n}$ presented in [240].

**Remark 2.** Assume that the projectors $A_{1\to n+1}$ (or $S_{1\to n+1}$) are equal to zero for some $n$, while $A_{1\to n}\neq 0\neq S_{1\to n}$. It leads to certain constraints on the parameter $\nu$. Indeed, from conditions $\kappa_{n+1}A_{1\to n+1}\kappa_{n+1}=0$ and $\kappa_{n+1}S_{1\to n+1}\kappa_{n+1}=0$ we obtain constraints $\kappa_{n+1}\sigma_n^{(+)}(q^n)\kappa_{n+1}=0$ and $\kappa_{n+1}\sigma_n^{(-)}(q^{-n})\kappa_{n+1}=0$, respectively. These constraints are equivalent to equations ($n>0$):

$$(\nu+q^{-(2n+1)})(\nu-q^{-(n-1)})(\nu+q^{-(n-1)})=0,$$
$$(\nu-q^{(2n+1)})(\nu-q^{(n-1)})(\nu+q^{(n-1)})=0.$$

It means that for $k>n$ all antisymmetrizers $A_{1\to k}$ could be equal to zero only if $\nu$ takes one of the values $\nu=-q^{-(2n+1)}, \pm q^{1-n}$, and, respectively, for $k>n$ all symmetrizers $S_{1\to k}$ could be equal to zero only if $\nu=q^{(2n+1)}, \pm q^{n-1}$. Recall (see Subsection **3.9**) that $\nu=q^{1-n}$ and $\nu=-q^{-1-2n}$ specify Birman–Murakami–Wenzl $R$-matrices for $SO_q(n)$ and $Sp_q(2(n+1))$ groups, respectively. The parameter $\nu=q^{n-1}$ could be related to the $Osp_q(2(m+1)-n|2m)$ $R$-matrix (3.11.52) with the choice (3.11.49), (3.11.50).

#### *4.4.3. Affine algebras $\alpha BMW_{M+1}$ and their central elements. Baxterized solution of reflection equation*

In Subsections **4.4.3** and **4.4.4**, we follow the presentation of the paper [242].

Affine Birman–Murakami–Wenzl algebras $\alpha BMW_{M+1}(q,\nu)$ are extensions of the algebras $BMW_{M+1}(q,\nu)$. The algebras $\alpha BMW_{M+1}$ are generated by the elements $\{\sigma_i,\kappa_i\}$ ($i=1,\dots,M$) with relations (4.1.1), (4.4.1)–(4.4.3) and the affine element $y_1$ which satisfies

$$\begin{gathered} \sigma_1\,y_1\,\sigma_1\,y_1 = y_1\,\sigma_1\,y_1\,\sigma_1, \quad [\sigma_k,\,y_1]=0 \quad \text{for}\ \ k>1, \\ \kappa_1\,y_1\,\sigma_1\,y_1\,\sigma_1 = c\,\kappa_1 = \sigma_1\,y_1\,\sigma_1\,y_1\,\kappa_1, \\ \kappa_1\,y_1^n\,\kappa_1 = \hat{z}^{(n)}\kappa_1, \quad n=1,2,3,\dots, \end{gathered} \tag{4.4.39}$$

where $c$, $\hat{z}^{(n)}$ are central elements. Initially, for the Brauer algebras, the affine version was introduced by M. Nazarov [245]. Below we use the set of affine elements

$$y_1, \quad y_{k+1}=\sigma_k\,y_k\,\sigma_k\ \in\ \alpha BMW_{M+1}, \quad k=1,2,\dots,M. \tag{4.4.40}$$

These elements generate a commutative subalgebra $Y_{M+1}$ in $\alpha BMW_{M+1}$.

We need some information about the center of $\alpha BMW$.

**Proposition 4.26.** *The elements*

$$\hat{\mathcal{Z}} = y_1 \cdot y_2 \cdots y_M, \quad \hat{\mathcal{Z}}_M^{(n)} = \sum_{k=1}^{M} \left( y_k^n - c^n \, y_k^{-n} \right), \quad n \in \mathbb{N} \tag{4.4.41}$$

*are central in the $\alpha BMW_M$ algebra.*

**Proof.** One can directly check the centrality of (4.4.41) by making use of the relations (4.4.39) and (4.4.40). ∎

**Remark 1.** The set of central "power sums" $\hat{\mathcal{Z}}^{(n)} = \sum_k (y_k^n - c^n \, y_k^{-n})$ is produced by the generating function

$$\mathcal{Z}(t) = \sum_{n=1} \hat{\mathcal{Z}}^{(n)} t^{n-1} = \frac{d}{dt} \log \left( \prod_{k=1} \frac{y_k - c\,t}{1 - y_k t} \right).$$

Consider an ascending chain of subalgebras

$$\alpha BMW_0 \subset \alpha BMW_1 \subset \alpha BMW_2 \subset \cdots \subset \alpha BMW_M \subset \alpha BMW_{M+1},$$

where $\alpha BMW_0, \alpha BMW_1$, and $\alpha BMW_j$ ($j > 1$) are, respectively, generated by $\{c, \hat{z}^{(n)}\}$, $\{c, \hat{z}^{(n)}, y_1\}$, and $\{c, \hat{z}^{(n)}, y_1, \sigma_1, \sigma_2, \dots, \sigma_{j-1}\}$. For the corresponding commutative subalgebras we have $Y_1 \subset Y_2 \subset \cdots \subset Y_M \subset Y_{M+1}$.

**Proposition 4.27.** *Let $\hat{Z}_k^{(n)}$ be central elements in the algebra $\alpha BMW_k$, $\alpha BMW_k \subset \alpha BMW_{k+2}$, defined by the relations*

$$\kappa_{k+1} y_{k+1}^n \kappa_{k+1} = \hat{Z}_k^{(n)} \, \kappa_{k+1} \in \alpha BMW_{k+2} \qquad (\hat{Z}_0^{(n)} \equiv \hat{z}^{(n)}, \quad \hat{Z}_k^{(0)} \equiv \hat{z}^{(0)} = \mu). \tag{4.4.42}$$

*Then the generating function for the elements $\hat{Z}_k^{(n)}$ is*

$$\begin{gathered} \sum_{n=0}^{\infty} \hat{Z}_k^{(n)} t^n = -\frac{\nu}{(q - q^{-1})} + \frac{1}{(1 - c\,t^2)} + \left( \sum_{n=0}^{\infty} t^n \, \hat{z}^{(n)} + \frac{\nu}{(q - q^{-1})} - \frac{1}{(1 - c\,t^2)} \right) \times \\ \times \prod_{r=1}^{k} \frac{(1 - y_r t)^2 (q^2 - c\, y_r^{-1} t)(q^{-2} - c\, y_r^{-1} t)}{(1 - c\, y_r^{-1} t)^2 (q^2 - y_r t)(q^{-2} - y_r t)}. \end{gathered} \tag{4.4.43}$$

**Proof.** We define the following function of central elements in $\alpha BMW_k$:

$$Q_k(t) = \sum_{n=0}^{\infty} \hat{Z}_k^{(n)} t^n + \frac{\nu}{(q - q^{-1})} - \frac{1}{(1 - c\,t^2)}.$$

Then one can deduce (see the method in [246]) the recursive formula

$$Q_k(t) = \frac{(1 - y_k t)^2 (q^2 - c\, y_k^{-1} t)(q^{-2} - c\, y_k^{-1} t)}{(1 - c\, y_k^{-1} t)^2 (q^2 - y_k t)(q^{-2} - y_k t)} \, Q_{k-1}(t), \tag{4.4.44}$$

where $Q_{k-1}(t) \in \alpha BMW_{k-1} \subset \alpha BMW_k$. From (4.4.44) we immediately obtain (4.4.43). ∎

**Remark 2.** The evaluation map $\alpha BMW_M \to BMW_M$ is defined by

$$y_1 \mapsto 1 \;\;\Rightarrow\;\; c \mapsto \nu^2, \quad \hat{z}^{(n)} \mapsto 1 + \frac{\nu^{-1} - \nu}{q - q^{-1}} \equiv \mu. \tag{4.4.45}$$

Under this map the function (4.4.43) transforms into the generating function presented in [246], where it is used for a proof of the Wenzl formula for the quantum dimensions of the $BMW_M$ primitive idempotents.

**Remark 3.** The homomorphisms of the periodic $\overline{BMW}_{M+1}$ algebra to the algebra $BMW_{M+1}$ and to the affine algebra $\alpha BMW_{M+1}$ are defined by the same Eqs. (4.3.62) and (4.3.72) as in the case of the group algebra of the braid group. Indeed, for the periodic $\overline{BMW}_{M+1}$ algebra the characteristic identity for $\sigma_M$ is the same as for $\sigma_1$, while the relations

$$\kappa_1 \sigma_M^{\pm 1} \kappa_1 = \nu^{\mp 1} \kappa_1, \quad \kappa_{M-1} \sigma_M^{\pm 1} \kappa_{M-1} = \nu^{\mp 1} \kappa_{M-1}$$

can be checked directly.

**Remark 4.** We redefine the Baxterized elements in (4.4.17), (4.4.21) as follows:

$$\sigma_i^{(a)}(x) = (\sigma_i - x\,\sigma_i^{-1}) + \frac{\lambda(\nu - a)}{(\nu - a\,x^{-1})}\,\kappa_i = (a^{-1} - a x^{-1})\frac{\sigma_i - a\,x}{\sigma_i - a\,x^{-1}}, \tag{4.4.46}$$

where we change the spectral parameter $x^2 \to x$ and denote by $a$ the solution of the equation $a^{-1} - a = \lambda \equiv q - q^{-1}$. It was discovered in [211] that the element of the affine BMW algebra

$$y_j(u) = f(u)\,\frac{y_j - \xi_a\,u}{y_j - \xi_a\,u^{-1}} \tag{4.4.47}$$

(here $\xi_a^2 := a\,c/\nu$ and $f(u)$ is any numerical function) solves the reflection equation (cf. (4.3.66), (5.2.3)):

$$y_j(u)\,\sigma_j\big(u\,v\big)\,y_j(v)\,\sigma_j\big(v\,u^{-1}\big) = \sigma_j\big(v\,u^{-1}\big)\,y_j(v)\,\sigma_j\big(u\,v\big)\,y_j(u). \tag{4.4.48}$$

This fact is important in the study of the evaluation homomorphisms for the quantum universal enveloping algebras; see [102] for the classical counterpart. The main ingredients of the fusion procedure [243] — the elements

$$\mathcal{Y}_j(u_1,\dots,u_{j-1},u) := \sigma_{j-1}(u\,u_{j-1})\,\mathcal{Y}_{j-1}(u_1,\dots,u_{j-2},u)\,\sigma_{j-1}(u\,u_{j-1}^{-1}), \qquad \mathcal{Y}_1(u) := y_1(u)$$

($j = 1,\dots,n-1$), also satisfy the reflection equation

$$\begin{aligned} &\mathcal{Y}_j(u_1,\dots,u_{j-1},u)\,\sigma_j\big(u\,v\big)\,\mathcal{Y}_j(u_1,\dots,u_{j-1},v)\,\sigma_j\big(v\,u^{-1}\big) = \\ &= \sigma_j\big(v\,u^{-1}\big)\,\mathcal{Y}_j(u_1,\dots,u_{j-1},v)\,\sigma_j\big(u\,v\big)\,\mathcal{Y}_j(u_1,\dots,u_{j-1},u). \end{aligned} \tag{4.4.49}$$

This is shown by induction in $j$.

*4.4.4. Intertwining operators in $\alpha BMW_{M+1}$ algebra*

Introduce the *intertwining* elements $U_{k+1} \in \alpha BMW_{M+1}$ ($k = 1,\dots,M$) (cf. (4.3.17)):

$$U_{k+1} = [\sigma_k,\, y_k - c\,y_{k+1}^{-1}]. \tag{4.4.50}$$

**Proposition 4.28.** *The elements $U_k$ satisfy (cf. (4.3.19)–(4.3.21)):*

$$U_{k+1}y_k = y_{k+1}\,U_{k+1}, \quad U_{k+1}y_{k+1} = y_k\,U_{k+1}, \quad U_{k+1}y_i = y_i\,U_{k+1} \quad \text{for} \quad i \neq k, k+1,$$

$$U_{k+1}\,[\sigma_k,\, y_k] = (qy_k - q^{-1}y_{k+1})(qy_{k+1} - q^{-1}y_k)\Big(1 - \frac{c}{y_k\, y_{k+1}}\Big), \tag{4.4.51}$$

$$U_{k+1}\,U_k\,U_{k+1} = U_k\,U_{k+1}\,U_k,$$

$$\kappa_k\,U_{k+1} = U_{k+1}\,\kappa_k = 0.$$

The elements $U_k$ provide an important information about the spectrum of the affine elements $\{y_j\}$ defined in (4.4.40).

**Lemma 3** (cf. Proposition **4.19**). *The spectrum of the elements $y_j \in \alpha BMW_{M+1}$ satisfies*

$$\mathrm{Spec}(y_j) \subset \{q^{2\mathbb{Z}} \cdot \mathrm{Spec}(y_1), \quad c\,q^{2\mathbb{Z}} \cdot \mathrm{Spec}(y_1^{-1})\}, \tag{4.4.52}$$

*where $\mathbb{Z}$ is the set of integer numbers.*

**Proof.** We prove it by induction in $j$. Equation (4.4.52) obviously holds for $y_1$. Assume that

$$\mathrm{Spec}(y_{j-1}) \subset \{q^{2\mathbb{Z}} \cdot \mathrm{Spec}(y_1), \;\; c\,q^{2\mathbb{Z}} \cdot \mathrm{Spec}(y_1^{-1})\}, \;\; j > 1.$$

Let $f$ be the characteristic polynomial of $y_{j-1}$, $f(y_{j-1}) = 0$. Then

$$0 = U_j f(y_{j-1})[\sigma_{j-1}, y_{j-1}] = f(y_j) U_j [\sigma_{j-1}, y_{j-1}] =$$
$$= f(y_j)(q^2 y_{j-1} - y_j)(y_j - q^{-2} y_{j-1})\left(y_j - c\,y_{j-1}^{-1}\right) y_j^{-1}.$$

Here we used (4.4.51). Thus, $\mathrm{Spec}(y_j) \subset \mathrm{Spec}(y_{j-1}) \cup q^{\pm 2} \cdot \mathrm{Spec}(y_{j-1}) \cup c \cdot \mathrm{Spec}(y_{j-1}^{-1})$. ∎

We denote the image of $w \in \alpha BMW_M$ under the evaluation map (4.4.45) by $\tilde{w}$, e.g., $y_j \mapsto \tilde{y}_j$. The Jucys–Murphy (JM) elements $\tilde{y}_j$ ($j = 2, \ldots, M$) defined in (4.4.28) are the images of $y_j$:

$$\tilde{y}_j = \sigma_{j-1} \ldots \sigma_2\, \sigma_1^2\, \sigma_2 \ldots \sigma_{j-1} \;\in\; BMW_M.$$

Lemma 3 provides the information about the spectrum of JM elements $\tilde{y}$'s.

**Corollary.** Since $\tilde{y}_1 = 1$ and $\tilde{c} = \nu^2$, it follows from (4.4.52) that

$$\mathrm{Spec}(\tilde{y}_j) \subset \{q^{2\mathbb{Z}},\; \nu^2 q^{2\mathbb{Z}}\}. \tag{4.4.53}$$

### *4.5. Representation theory of $BMW_{M+1}$ algebras*

The representation theory for the Birman–Murakami–Wenzl algebra was constructed in [244] (see also [247, 248]). The approach considered in this subsection (the colored Young graph, the analog of Proposition **4.20**, the explicit formulas for all primitive idempotents in terms of the Jucys–Murphy elements, intertwiner operators $U_k$ (4.3.19)–(4.3.21), etc.) similar to that presented for the Hecke algebra case in Subsection **4.3** was developed in [242].

*4.5.1. Representations of affine algebra $\alpha BMW_2$*

**A. $\alpha BMW_2$ algebra and its modules $V_D$**

The elements $\{y_i, y_{i+1}, \sigma_i, \kappa_i\} \in \alpha BMW_M$ (for fixed $i < M$) satisfy

$$(q - q^{-1})\kappa_i = \sigma_i^{-1} - \sigma_i + (q - q^{-1}), \tag{4.5.1}$$

$$y_{i+1} = \sigma_i y_i \sigma_i, \quad y_i y_{i+1} = y_{i+1} y_i, \quad \kappa_i y_i^n \kappa_i = \hat{Z}_{i-1}^{(n)} \kappa_i, \tag{4.5.2}$$

$$y_i y_{i+1} \kappa_i = c\, \kappa_i = \kappa_i y_{i+1} y_i. \tag{4.5.3}$$

The elements $c$ and $\hat{Z}_{i-1}^{(n)}$ commute with $\{y_i, y_{i+1}, \sigma_i, \kappa_i\}$. The elements $\{y_i, y_{i+1}, \sigma_i, \kappa_i\} \in \alpha BMW_M$ generate a subalgebra isomorphic to $\alpha BMW_2$.

Below we investigate representations $\rho$ of $\alpha BMW_2$ for which the generators $\rho(y_i)$ and $\rho(y_{i+1})$ are diagonalizable and $\rho(c) = \nu^2 \cdot \mathrm{Id}$. Let $\psi$ be a common eigenvector of $\rho(y_i)$ and $\rho(y_{i+1})$ with some eigenvalues $a$ and $b$:

$$\rho(y_i)\, \psi = a\, \psi, \quad \rho(y_{i+1})\, \psi = b\, \psi.$$

The element $\hat{z} = y_i y_{i+1}$ is central in $\alpha BMW_2$. There are two possibilities:

$$\begin{array}{ll} \mathbf{1.} & \rho(\kappa_i) \neq 0 \quad \overset{\text{Eq. (4.5.3)}}{\Longrightarrow} \quad \rho(y_i y_{i+1}) = \nu^2 \cdot \mathrm{Id} \;\; \Rightarrow \;\; \underline{a\, b = \nu^2}; \\ \mathbf{2.} & \rho(\kappa_i) = 0, \;\; \underline{\text{the product } a\, b \text{ is not fixed}}. \end{array} \tag{4.5.4}$$

Further for brevity we often omit the symbol $\rho$ and denote the operator $\rho(x)$ for $x \in \alpha BMW$ by the same letter $x$; this should not lead to a confusion.

Applying the operators from $\alpha BMW_2$ to the vector $\psi$, we produce, in general, infinite-dimensional $\alpha BMW_2$-module $V_\infty$ spanned by

$$\begin{array}{ll} & e_2 = \psi, \\ e_1 = \kappa_i \psi, & e_3 = \sigma_i \psi, \\ e_4 = y_i \kappa_i \psi, & e_5 = \sigma_i y_i \kappa_i \psi, \\ e_6 = y_i^2 \kappa_i \psi, & e_7 = \sigma_i y_i^2 \kappa_i \psi, \\ \quad \dots\dots\dots, & \quad \dots\dots\dots, \\ e_{2k+2} = y_i^k \kappa_i \psi, & e_{2k+3} = \sigma_i y_i^k \kappa_i \psi \;\; (k \geqslant 1), \dots\dots\dots \end{array}$$

Using relations (4.5.1)–(4.5.3) for $\alpha BMW_2$, one can write down the left action of elements $\{y_i, y_{i+1}, \sigma_i, \kappa_i\}$ on $V_\infty$. Our aim is to understand when the sequence $e_j$ can terminate giving therefore rise to a finite-dimensional module $V_D$ (of dimension $D$) of $\alpha BMW_2$ and to investigate the (ir)reducibility of $V_D$.

We distinguish three cases for the module $V_D$:

**(i)** $\kappa_i V_D = 0$ (i.e., $\kappa_i\, e = 0 \;\; \forall\, e \in V_D$) and, in particular, $\kappa_i\, \psi = 0$. Therefore, $e_j = 0$ for all $j \neq 2, 3$ and $V_\infty$ reduces to a 2-dimensional module with the basis $\{e_2, e_3\}$. In view of (4.5.4), the product $a\, b$ is not fixed and the irreps coincide with the irreps of the affine Hecke algebra $\alpha H_2$ considered in Subsection **4.3.3** and in [208, 209].

**(ii)** $\kappa_i V_D \neq 0$ (i.e., $\exists\, e \in V_D$: $\kappa_i e \neq 0$). The module $V_D$ is extracted from $V_\infty$ by constraints

$$e_{2k+4} = \sum_{m=1}^{2k+3} \alpha_m\, e_m \quad (k \geqslant -1), \;\; ab = \nu^2, \tag{4.5.5}$$

with some parameters $\alpha_m$. The independent basis vectors are $(e_1, e_2, \dots, e_{2k+3})$. The module $V_D$ has <u>odd dimension</u>.

**(iii)** $\kappa_i V_D \neq 0$ and additional constraints are

$$e_{2k+3} = \sum_{m=1}^{2k+2} \alpha_m \, e_m \quad (k \geqslant 0), \;\; ab = \nu^2. \tag{4.5.6}$$

The independent basis vectors are $(e_1, e_2, \dots, e_{2k+2})$. The module $V_D$ has even dimension.

Below we consider a version $\alpha' BMW_2$ of the affine BMW algebra. The additional requirement for this algebra concerns the spectrum of $y_i, y_{i+1} \in \alpha' BMW_2$:

$$\mathrm{Spec}(y_j) \subset \{q^{2\mathbb{Z}}, \; \nu^2 q^{2\mathbb{Z}}\}.$$

The evaluation map (4.4.45) descends to the algebra $\alpha' BMW$ (cf. Corollary after Lemma 3). In particular, for the cases **(ii)** and **(iii)** we have

$$a = \nu^2 q^{2z}, \quad b = q^{-2z} \quad \text{or} \quad a = q^{2z}, \quad b = \nu^2 q^{-2z}$$

for some $z \in \mathbb{Z}$.

**B. The case $\kappa_i V_D = 0$: Hecke algebra case [208, 209] (see also Subsection 4.3.3).**

Representations of $\alpha BMW_2$ with $\kappa_i V_D = 0$ reduce to representations of the affine Hecke algebra $\alpha H_2$. In the basis of two vectors $(e_2, e_3) = (\psi, \sigma_i \psi)$, the matrices of the generators are (cf. (4.3.32)):

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}, \; y_i = \begin{pmatrix} a & -(q-q^{-1})b \\ 0 & b \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} b & (q-q^{-1})b \\ 0 & a \end{pmatrix}, \tag{4.5.7}$$

where $a \neq b$ (otherwise $y_i, y_{i+1}$ are not diagonalizable). By Lemma 3, we have for $y_i, y_{i+1} \in \alpha' BMW_2$ the eigenvalues $a, b \in \{q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}}\}$. The 2-dimensional representation (4.5.7) contains a 1-dimensional subrepresentation iff $a = q^{\pm 2} b$. Graphically these 1- and 2-dimensional irreps of $\alpha' BMW_2$ are visualized by the same pictures as in Figures 4.2 and 4.3 in Subsection **4.3.3**. Different paths going from the upper vertex to the lower vertex correspond to different eigenvectors of $y_i, y_{i+1}$. The indices on the edges are eigenvalues of $y_i, y_{i+1}$.

**C. $\kappa_i V_D \neq 0$: odd-dimensional representations for $\alpha' BMW_2$**

Using condition (4.5.5) for the reduction $V_\infty$ to $V_{2m+1}$, one can describe odd-dimensional representations of $\alpha' BMW_2$, determine matrices for the action of $y_i$, $y_{i+1}$ on $V_{2m+1}$ and calculate

$$\det(y_i) = \prod_{r=1}^{2m+1} y_i^{(r)} = \nu^{2m}, \quad \det(y_{i+1}) = \prod_{r=1}^{2m+1} y_{i+1}^{(r)} = \nu^{2m+2}. \tag{4.5.8}$$

Here for eigenvalues $y_i^{(r)}$, $y_{i+1}^{(r)}$ $(r = 1, 2, \dots, 2m+1)$ of $y_i$ and $y_{i+1}$ we have constraints

$$y_i^{(r)} \, y_{i+1}^{(r)} = \nu^2, \quad r = 1, \dots, 2m+1$$

and (see Eq. (4.4.53))

$$y_i^{(r)} \in \{q^{2\mathbb{Z}}, \; \nu^2 q^{2\mathbb{Z}}\}, \;\; r = 1, \dots, 2m+1.$$

These odd-dimensional irreps are visualized as graphs:

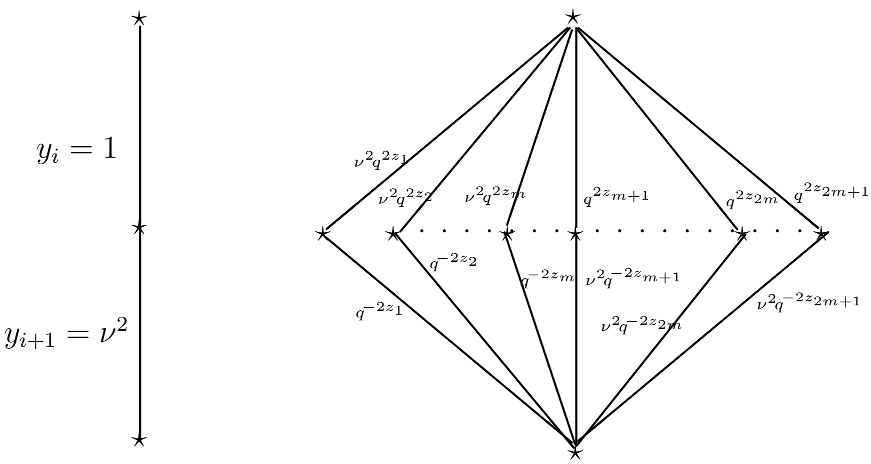
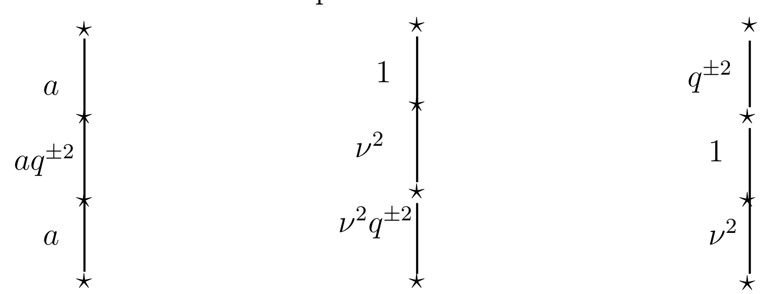



**Figure 4.4**

where $z_r \in \mathbb{Z}$ and $\sum_{r=1}^{2m+1} z_r = 0$, as it follows from (4.5.8). Different paths going from the top vertex to the bottom vertex correspond to different common eigenvectors of $y_i, y_{i+1}$. Indices on upper and lower edges of these paths are the eigenvalues of $y_i$ and $y_{i+1}$, respectively.

**Remark.** In view of the braid relations $\sigma_i \sigma_{i\pm1} \sigma_i = \sigma_{i\pm1} \sigma_i \sigma_{i\pm1}$ and possible eigenvalues of $\sigma$'s for 1-dimensional representations (described in Subsections **3.2** and **3.3**), we conclude that the following chains of 1-dimensional representations are forbidden:




where $a = q^{2z}$ or $a = \nu^2 q^{2z}$ $(z \in \mathbb{Z})$.

**D.** $\kappa_i V_D \neq 0$**: even-dimensional representations of** $\alpha' BMW_2$

With the help of conditions (4.5.6) we reduce $V_\infty$ to $V_{2m}$, then explicitly construct $(2m) \times (2m)$ matrices for the operators $y_i, y_{i+1}$ and calculate their determinants

$$\det(y_i) = \prod_{r=1}^{2m} y_i^{(r)} = \epsilon q^{\epsilon}\, \nu^{2m-1}, \quad \det(y_{i+1}) = \prod_{r=1}^{2m} y_{i+1}^{(r)} = -\epsilon q^{\epsilon}\, \nu^{2m+1}, \tag{4.5.9}$$

where $y_i^{(r)}$, $y_{i+1}^{(r)}$ are eigenvalues of $y_i$, $y_{i+1}$ (we have two possibilities: $\epsilon = \pm1$). We see from (4.5.9) that all $(2m)$ eigenvalues of $y_i$, $y_{i+1}$ cannot belong to the spectrum (4.4.53). More precisely, there is at least one eigenvalue $y_i^{(r)}$ of $y_i$ (and the eigenvalue $y_{i+1}^{(r)}$ of $y_{i+1}$) such that

$$y_i^{(r)}, y_{i+1}^{(r)} \notin \{q^{2\mathbb{Z}},\ \nu^2 q^{2\mathbb{Z}}\}.$$

Thus, even-dimensional irreps of $\alpha BMW_2$ subject to the conditions (4.5.6) are not admissible for $\alpha' BMW_2$.

*4.5.2.* Spec$(y_1,\ldots,y_n)$ *and rules for strings of eigenvalues*

Now we reconstruct the representation theory of BMW algebras using an approach which generalizes the approach of Okounkov–Vershik [231] for symmetric groups.

The JM elements $\{\tilde{y}_1,\ldots,\tilde{y}_n\}$ generate a commutative subalgebra in $BMW_n$. The basis in the space of an irrep of $BMW_n$ can be chosen to be the common eigenbasis of all $\tilde{y}_i$. Each common eigenvector $v$ of $\tilde{y}_i$,

$$\tilde{y}_i\, v = a_i\, v, \quad i = 1,\ldots,n,$$

defines a string $(a_1,\ldots,a_n) \in \mathbb{C}^n$. Denote by Spec$(\tilde{y}_1,\ldots,\tilde{y}_n)$ the set of such strings.

We summarize our results about representations of $\alpha' BMW_2$ and the spectrum of the JM elements $\tilde{y}_i$ in the following Proposition.

**Proposition 4.29.** *Consider the string*

$$\alpha = (a_1,\ldots,a_i,a_{i+1},\ldots,a_n) \in \mathrm{Spec}(\tilde{y}_1,\ldots,\tilde{y}_i,\tilde{y}_{i+1},\ldots,\tilde{y}_n).$$

*Let* $v_\alpha$ *be the corresponding eigenvector of* $\tilde{y}_i$*:* $\tilde{y}_i\, v_\alpha = a_i\, v_\alpha$*. Then*

**(1)** $a_i \in \{q^{2\mathbb{Z}},\ \nu^2 q^{2\mathbb{Z}}\}$ ;
**(2)** $a_i \neq a_{i+1}, \ \ i = 1,\ldots,n-1$ ;
**(3a)** $a_i a_{i+1} \neq \nu^2, \quad a_{i+1} = q^{\pm 2}\, a_i \Rightarrow \sigma_i \cdot v_\alpha = \pm q^{\pm 1} v_\alpha, \ \ \kappa_i \cdot v_\alpha = 0$ ;
**(3b)** $a_i a_{i+1} \neq \nu^2, \quad a_{i+1} \neq q^{\pm 2}\, a_i \Rightarrow$
$\alpha' = (a_1,\ldots,a_{i+1},a_i,\ldots,a_n) \in \mathrm{Spec}(\tilde{y}_1,\ldots,\tilde{y}_i,\tilde{y}_{i+1},\ldots,\tilde{y}_n), \ \kappa_i \cdot v_\alpha = 0, \ \kappa_i \cdot v_{\alpha'} = 0$ ;
**(4)** $a_i a_{i+1} = \nu^2 \ \Rightarrow \ \exists$ odd number of strings $\alpha^{(k)}$ $(k = 1,2,\ldots,2m+1)$ :

$$\alpha^{(k)} = (a_1,\ldots,a_{i-1},a_i^{(k)},a_{i+1}^{(k)},a_{i+2},\ldots,a_n) \in \mathrm{Spec}(\tilde{y}_1,\ldots,\tilde{y}_n) \ \ \forall k,$$

$$\alpha \in \{\alpha^{(k)}\}, \ \ a_i^{(k)} a_{i+1}^{(k)} = \nu^2, \ \ \prod_{k=1}^{2m+1} a_i^{(k)} = \nu^{2m}, \ \ \prod_{k=1}^{2m+1} a_{i+1}^{(k)} = \nu^{2m+2}.$$

The necessary and sufficient conditions for a string to belong to the common spectrum of $\tilde{y}_i$ are formulated in the following way.

**Proposition 4.30.** *The string* $\alpha = (a_1,a_2,\ldots,a_n)$*, where* $a_i \in (q^{2\mathbb{Z}},\nu^2 q^{2\mathbb{Z}})$*, belongs to the set* Spec$(\tilde{y}_1,\tilde{y}_2,\ldots,\tilde{y}_n)$ *iff* $\alpha$ *satisfies the following conditions* ($z \in \mathbb{Z}$)*:*

**(1)** $a_1 = 1$ ; **(2)** $a_i = \nu^2 q^{-2z} \ \Rightarrow q^{2z} \in \{a_1,\ldots,a_{i-1}\}$ ;

**(3)** $a_i = q^{2z} \Rightarrow \{a_i q^2, a_i q^{-2}\} \cap \{a_1,\ldots,a_{i-1}\} \neq \varnothing, \ \ z \neq 0$;

**(4a)** $a_i = a_j = q^{2z}\ (i<j) \Rightarrow \begin{cases} \text{either } \{q^{2(z+1)}, q^{2(z-1)}\} \subset \{a_{i+1},\ldots,a_{j-1}\}, \\ \text{or } \nu^2 q^{-2z} \in \{a_{i+1},\ldots,a_{j-1}\} \ ; \end{cases}$

**(4b)** $a_i = a_j = \nu^2 q^{2z}\ (i<j) \Rightarrow \begin{cases} \text{either } \{\nu^2 q^{2(z+1)}, \nu^2 q^{2(z-1)}\} \subset \{a_{i+1},\ldots,a_{j-1}\}, \\ \text{or } q^{-2z} \in \{a_{i+1},\ldots,a_{j-1}\} \ ; \end{cases}$

**(5a)** $a_i = \nu^2 q^{-2z}, \ \ a_j = q^{2z'}\ (i<j) \Rightarrow q^{2z} \text{ or } \nu^2 q^{-2z'} \in \{a_{i+1},\ldots,a_{j-1}\}$ ;

**(5b)** $a_i = q^{2z}, \ \ a_j = \nu^2 q^{-2z'}\ (i<j) \Rightarrow \nu^2 q^{-2z} \text{ or } q^{2z'} \in \{a_{i+1},\ldots,a_{j-1}\}$.

*where in* **(5a)** *and* **(5b)** *we set* $z' = z \pm 1$.

*4.5.3. Colored Young graph for BMW algebras*

We illustrate the above considerations on the example of the colored (in the sense of [208, 209]) Young graph for the algebra $BMW_5$. This graph contains the whole information about the irreps of $BMW_5$ and the branching rules $BMW_5 \downarrow BMW_4$.

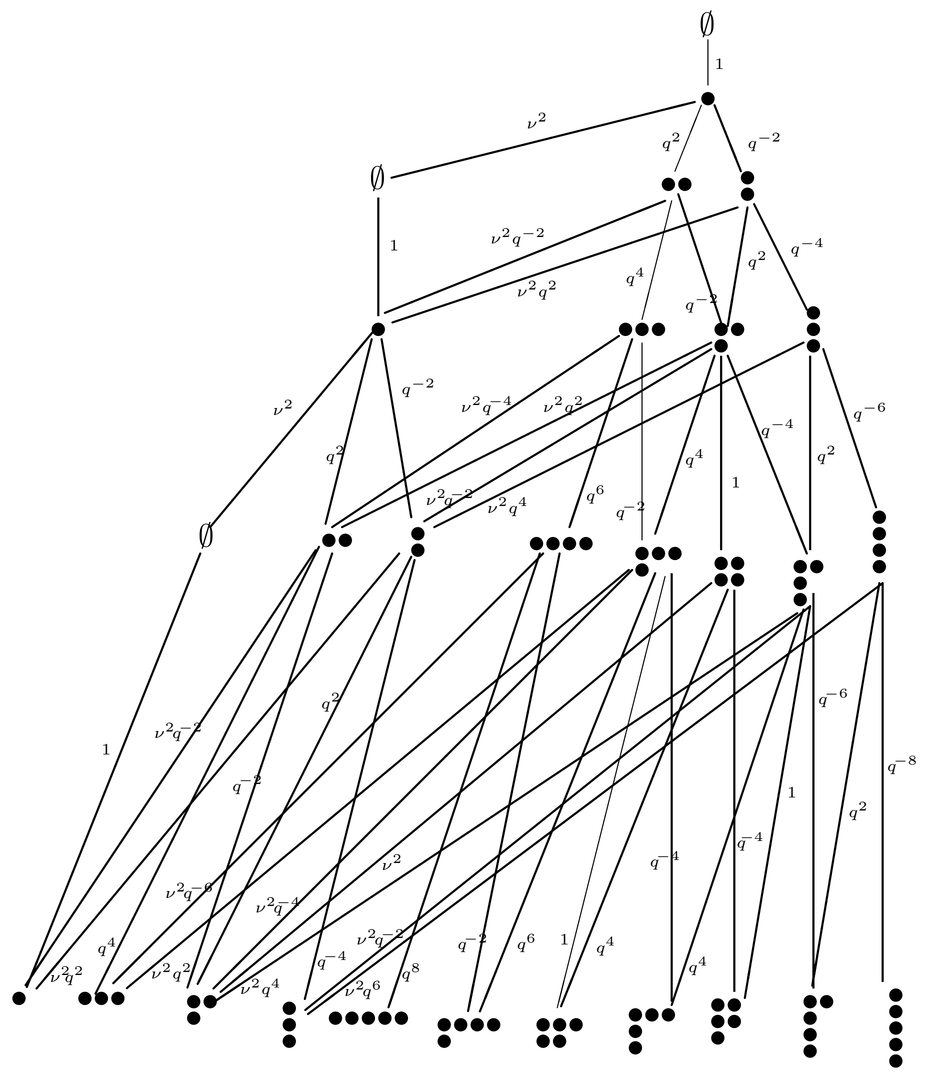
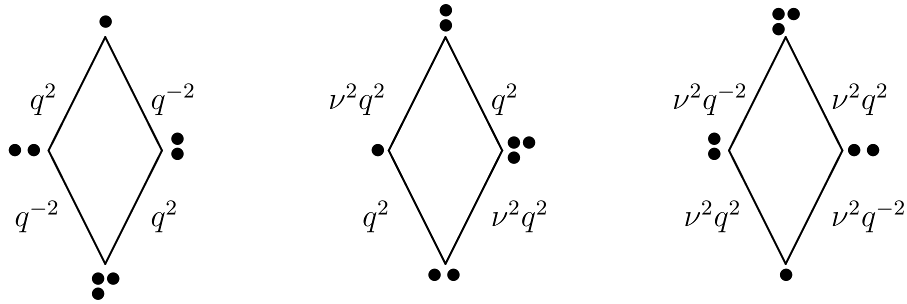



**Figure 4.5**

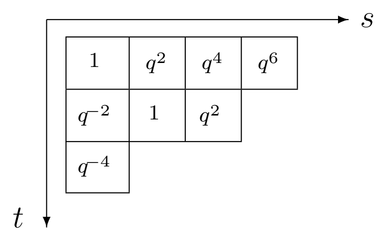
A vertex $\{\lambda; 5\}$ on the lowest level of this graph is labeled by some Young diagram $\lambda$; this vertex corresponds to the irrep $W_{\{\lambda;5\}}$ of $BMW_5$ (the notation $\{\lambda; 5\}$ is designed to encode the diagram $\lambda$ and the level on which this diagram is located; the levels are counted starting from 0). Paths going down from the top vertex $\emptyset$ to the lowest level (that is, paths of length 5) correspond to common eigenvectors of the JM elements $\tilde{y}_1, \ldots, \tilde{y}_5$. Paths ending at $\{\lambda; 5\}$ label the basis in $W_{\{\lambda;5\}}$. In particular, the number of different paths going down from the top $\emptyset$ to $\{\lambda; 5\}$ is equal to the dimension of the irrep $W_{\{\lambda;5\}}$.

Note that the colored Young graph in Figure 4.5 contains subgraphs presented in Figures 4.2, 4.3 and 4.4. For example, in Figure 4.5 one recognizes rhombic subgraphs (the vertices on the subgraphs are obtained from one another by a rotation)

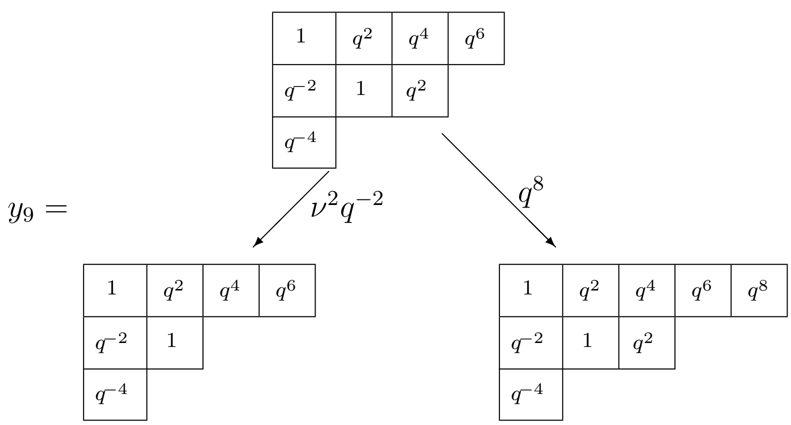



of the type presented in Figure 4.3.

Let $(s, t)$ be coordinates of a node in the Young diagram $\lambda$. To the node $(s, t)$ of the diagram $\lambda$ we associate a number $q^{2(s-t)}$ which is called “content”:




Then, according to the colored Young graph in Figure 4.5, at each step down along the path one can add or remove one node (therefore, this graph is called the “oscillating” Young graph) and the eigenvalue of the corresponding JM element is determined by the content of the node:




The eigenvalue corresponding to the addition or removal of the $(s, t)$ node is $q^{2(s-t)}$ or $\nu^2 q^{-2(s-t)}$, respectively.

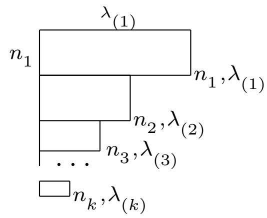
Let $X(n)$ be the set of paths of length $n$ starting from the top vertex $\emptyset$ and going down in the Young graph of oscillating Young diagrams. Now we formulate the following Proposition.

**Proposition 4.31 .** *There is a bijection between the set* $\mathrm{Spec}(\tilde{y}_1, \ldots, \tilde{y}_n)$ *and the set* $X(n)$.

### *4.5.4. Primitive idempotents*

The colored Young graph (as in Figure 4.5) gives also the rule of construction of a complete set of orthogonal primitive idempotents for the BMW algebra. The completeness of the set of orthogonal primitive idempotents is equivalent to the maximality of the commutative set of JM elements. Let $\{\Lambda; n\}$ be a vertex in the Young graph with

$\Lambda \;=$




$(n_i, \lambda_{(i)})$ are coordinates of the nodes which are in the corners of the diagram $\Lambda = [\lambda_{(1)}^{n_1}, \lambda_{(2)}^{n_2-n_1}, \dots, \lambda_{(k)}^{n_k-n_{k-1}}]$ . (4.5.10)

Consider any path $T_{\{\Lambda;n\}}$ going down from the top $\emptyset$ to this vertex. Let $E_{T_{\{\Lambda;n\}}} \in BMW_n$ be the primitive idempotent corresponding to $T_{\{\Lambda;n\}}$. Using the branching rule implied by the Young graph for $BMW_{n+1}$, we know all possible eigenvalues of the element $\tilde{y}_{n+1}$ and, therefore, obtain the identity

$$E_{T_{\{\Lambda;n\}}} \cdot \prod_{r=1}^{k+1} \big(\tilde{y}_{n+1} - q^{2(\lambda_{(r)}-n_{r-1})}\big) \prod_{r=1}^{k} \big(\tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda_{(r)})}\big) = 0,$$

where $\lambda_{(k+1)} = n_0 = 0$. So, for a new diagram $\Lambda'$ obtained by adding to $\Lambda$ a new node with coordinates $(n_{j-1}+1, \lambda_{(j)}+1)$ the corresponding primitive idempotent (after an appropriate normalization) reads

$$E_{T_{\{\Lambda';n+1\}}} = E_{T_{\{\Lambda;n\}}} \cdot \prod_{\substack{r=1\\ r\neq j}}^{k+1} \frac{\big(\tilde{y}_{n+1} - q^{2(\lambda_{(r)}-n_{r-1})}\big)}{\big(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(r)}-n_{r-1})}\big)} \prod_{r=1}^{k} \frac{\big(\tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda_{(r)})}\big)}{\big(q^{2(\lambda_{(j)}-n_{j-1})} - \nu^2 q^{2(n_r-\lambda_{(r)})}\big)} \,.$$

For a new diagram $\Lambda''$ which is obtained from $\Lambda$ by removing a node with coordinates $(n_j, \lambda_{(j)})$ we construct the primitive idempotent

$$E_{T_{\{\Lambda'';n+1\}}} = E_{T_{\{\Lambda;n\}}} \cdot \prod_{r=1}^{k+1} \frac{\big(\tilde{y}_{n+1} - q^{2(\lambda_{(r)}-n_{r-1})}\big)}{\big(\nu^2 q^{2(n_j-\lambda_{(j)})} - q^{2(\lambda_{(r)}-n_{r-1})}\big)} \prod_{\substack{r=1\\ r\neq j}}^{k} \frac{\big(\tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda_{(r)})}\big)}{\big(\nu^2 q^{2(n_j-\lambda_{(j)})} - \nu^2 q^{2(n_r-\lambda_{(r)})}\big)} \,.$$

Using these formulas and the "initial data" $E_{T_{\{\emptyset;0\}}} = 1$, one can deduce step by step explicit expressions for the primitive orthogonal idempotents related to the paths in the BMW Young graph.

**Remark.** In this subsection, we reconstructed the representation theory of the tower of the BMW algebras using the properties of the commutative subalgebras generated by the Jucys–Murphy elements in the BMW algebras. This representation theory is of use in the representation theory of the quantum groups $U_q(osp(N|K))$ due to the Brauer–Schur–Weyl duality, but also finds applications in physical models. Recently [211], we have formulated integrable chain models with nontrivial boundary conditions in terms of the affine Hecke algebras $\hat{H}_n$ and the affine BMW algebras $\alpha BMW_n$. The Hamiltonians for these models are special elements of the algebras $\hat{H}_n$ and $\alpha BMW_n$. For example, for the $\alpha BMW_n$ algebra we deduced [211] the Hamiltonians

$$\mathcal{H} = \sum_{m=1}^{n-1} \left( \sigma_m + \frac{(q-q^{-1})\nu}{\nu + a} \kappa_m \right) + \frac{(q-q^{-1})\xi}{y_1 - \xi}, \qquad (4.5.11)$$

where $\xi^2 = -a\,c/\nu$ and the parameter $a$ can take one of two values $a = \pm q^{\pm 1}$. Now different representations $\rho$ of the algebra $\alpha BMW_n$ give different integrable spin chain models with Hamiltonians $\rho(\mathcal{H})$ which, in particular, possess $U_q(osp(N|K))$ symmetries for some $N$ and $K$. So, the representations $\rho$ of the algebra $\alpha BMW_n$ are related to the spin chain models of *osp* type with $n$ sites and nontrivial boundary conditions. The BMW chains (chains based on the BMW algebras in the $R$-matrix representations) describe in a unified way spin chains with $U_q(osp(N|K))$ symmetries.

The Hamiltonians for the Hecke chain models are obtained from the Hamiltonians for the BMW chain models by taking the quotient $\kappa_j = 0$. These models were considered in [212, 213]. The Hecke chains (chain models based on the Hecke algebras) describe in a unified way spin chains with $U_q(sl(N|K))$ symmetries. In [212–214], we investigated the integrable open chain models formulated in terms of the generators of the Hecke algebra (non-affine case, $y_1 = 1$). For the open Hecke chains of a finite size, the spectrum of the Hamiltonians with free boundary conditions is determined [213] for special (corner type) irreducible representations of the Hecke algebra. In [212], we investigated the functional equations for the transfer matrix type elements of the Hecke algebra appearing in the theory of Hecke chains.

*4.5.5. q-Dimensions of idempotents in the BMW algebra*

Consider the following inclusions of the subalgebras $\alpha BMW_1 \subset \alpha BMW_2 \subset \cdots \subset \alpha BMW_{n+1}$:

$$\{y_1; \sigma_1, \dots, \sigma_{k-1}\} \in \alpha BMW_k \subset \alpha BMW_{k+1} \ni \{y_1; \sigma_1, \dots, \sigma_{k-1}, \sigma_k\}.$$

For the subalgebras $\alpha BMW_{k+1}$ we introduce linear mapping (quantum trace)

$$\mathrm{Tr}_{(k+1)}: \quad \alpha BMW_{k+1} \to \alpha BMW_k, \quad (k = 1, 2, \dots, n),$$

which is defined by the formula (cf. (3.10.39))

$$\kappa_{k+1}\, X_{k+1}\, \kappa_{k+1} = \frac{1}{\nu}\, \mathrm{Tr}_{(k+1)}(X_{k+1})\, \kappa_{k+1}, \quad \forall X_{k+1} \in \alpha BMW_{k+1}. \tag{4.5.12}$$

**Proposition 4.32** *(see [187]). For the map* $\mathrm{Tr}_{(k+1)}$*:* $\alpha BMW_{k+1} \to \alpha BMW_k$ *we have the following properties* ($\forall X_k, X'_k \in \alpha BMW_k$, $\forall Y_{k+1} \in \alpha BMW_{k+1}$)*:*

$$\begin{gathered} \mathrm{Tr}_{(k+1)}(\sigma_k) = 1, \;\; \mathrm{Tr}_{(k+1)}(\sigma_k^{-1}) = \nu^2, \;\; \mathrm{Tr}_{(k+1)}(X_k) = \nu\, \mu\, X_k, \\ \mathrm{Tr}_{(k+1)}(\kappa_k) = \nu, \;\; \mathrm{Tr}_{(1)}(y_1^k) = \nu\, \hat{z}^{(k)}, \;\; \mathrm{Tr}_{(1)}(1) = \nu\, \mu = (1 + \lambda\nu - \nu^2)\lambda^{-1}, \end{gathered} \tag{4.5.13}$$

$$\mathrm{Tr}_{(k+1)}(\sigma_k\, X_k\, \sigma_k^{-1}) = \mathrm{Tr}_{(k)}(X_k) = \mathrm{Tr}_{(k+1)}(\sigma_k^{-1}\, X_k\, \sigma_k), \tag{4.5.14}$$

$$\mathrm{Tr}_{(k+1)}(\sigma_k\, X_k\, \kappa_k) = \mathrm{Tr}_{(k+1)}(\kappa_k\, X_k\, \sigma_k), \tag{4.5.15}$$

$$\begin{gathered} \mathrm{Tr}_{(k+1)}(X_k \cdot Y_{k+1} \cdot X'_k) = X_k \cdot \mathrm{Tr}_{(k+1)}(Y_{k+1}) \cdot X'_k \;\; , \\ \mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(\sigma_k \cdot Y_{k+1}) = \mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(Y_{k+1} \cdot \sigma_k). \end{gathered} \tag{4.5.16}$$

By using the mapping (4.5.12), definitions (4.4.42), (4.4.39) and evaluation (4.4.45), we write relation (4.4.43) in the form (cf. (4.3.75))

$$\begin{gathered} \lambda \mathrm{Tr}_{_{(M+1)}}\Big(\frac{y_{M+1}}{t - y_{M+1}}\Big) + 1 - \frac{\lambda\nu^3}{(t^2 - \nu^2)} = \\ = \frac{(t - \nu^2)(t - q^{-1}\nu)(t + q\nu)}{(t - 1)(t - \nu)(t + \nu)} \cdot \prod_{r=1}^{M} \frac{(t - y_r)^2 (q^2 t - \nu^2\, y_r^{-1})(q^{-2} t - \nu^2\, y_r^{-1})}{(t - \nu^2\, y_r^{-1})^2 (q^2 t - y_r)(q^{-2} t - y_r)}, \end{gathered} \tag{4.5.17}$$

where we change variable $t \to t^{-1}$, index $k \to M$ and for simplicity denote $y_r = \tilde{y}_r$. Then we act to both sides of (4.5.17) by the idempotent $E_{T_{\{\Lambda,M\}}}$, where $T_{\{\Lambda,M\}}$ is the path of length $M$

in the colored Young graph (of the type presented in Figure 4.5) with the final vertex labeled by the Young diagram $\Lambda$ (4.5.10). According to the branching rule, which is implied by the colored Young graph for $BMW_{M+1}$, we use the expansion

$$E_{T_{\{\Lambda,M\}}} = \sum_{j=1}^{k+1} E_{T_{\{\Lambda'_j,M+1\}}} + \sum_{j=1}^{k} E_{T_{\{\Lambda''_j,M+1\}}},$$

where the Young diagram $\Lambda'_j$ is obtained by adding a node to the outer corners $(n_{j-1}+1, \lambda_{(j)}+1)|_{j=1,\dots,k+1}$ of the diagram $\Lambda$ and the diagram $\Lambda''_j$ is obtained by removing a node from inner corners $(n_j, \lambda_{(j)})|_{j=1,\dots,k}$ of $\Lambda$. As a result, we obtain

$$\begin{gathered}\mathrm{Tr}_{(M+1)}\Big(\lambda \frac{y_{M+1}}{t-y_{M+1}}\Big(\sum_{j=1}^{k+1} E_{T_{\{\Lambda'_j,M+1\}}} + \sum_{j=1}^{k} E_{T_{\{\Lambda''_j,M+1\}}}\Big)\Big) = \\ = \Big(\frac{(t-\nu^2)(t-q^{-1}\nu)(t+q\nu)}{(t-1)(t-\nu)(t+\nu)} \cdot \prod_{r=1}^{M} \frac{(t-y_r)^2(t-q^{-2}\nu^2\, y_r^{-1})(t-q^2\nu^2\, y_r^{-1})}{(t-q^{-2}y_r)(t-q^2 y_r)(t-\nu^2\, y_r^{-1})^2} - c(t)\Big) E_{T_{\{\Lambda,M\}}},\end{gathered} \tag{4.5.18}$$

where $c(t) := 1 - \frac{\lambda\nu^3}{(t^2-\nu^2)}$. Now we note that, if a cell with content $q^{2a}$ was added on the step $i$ in the path $T_{\{\Lambda,M\}}$ and further this cell was removed on the step $j > i$ (it means that $y_i = q^{2a}$ and $y_j = \nu^2 q^{-2a}$), then the factors with $r = i$ and $r = j$ are canceled in the product in the r.h.s. of (4.5.18). Thus, the only factors contribute in this product that correspond to adding cells to form the diagram $\Lambda$ . In this case, we can substitute the eigenvalues (4.3.81) of $y_r$ and consider $M$ as a number of cells in the diagram $\Lambda \vdash M$. After a cancelation of many factors in the r.h.s. of (4.5.18) (see the derivation of (4.3.82)) we write (4.5.18) in the form

$$\begin{gathered}\sum_{j=1}^{k+1}\Big(\frac{\lambda q^{2(\lambda_{(j)}-n_{j-1})}}{t-q^{2(\lambda_{(j)}-n_{j-1})}}\Big)\mathrm{Tr}_{(M+1)} E_{T_{\{\Lambda'_j,M+1\}}} + \sum_{j=1}^{k}\Big(\frac{\lambda\nu^2 q^{2(n_j-\lambda_{(j)})}}{t-\nu^2 q^{2(n_j-\lambda_{(j)})}}\Big)\mathrm{Tr}_{(M+1)} E_{T_{\{\Lambda''_j,M+1\}}} = \\ = \Big(\frac{(t-\nu^2 q^{2n})(t-q^{-1}\nu)(t+q\nu)}{(t-q^{-2n})(t-\nu)(t+\nu)} \cdot \prod_{r=1}^{k} \frac{(t-q^{2(\lambda_{(r)}-n_r)})(t-\nu^2\, q^{2(n_{r-1}-\lambda_{(r)})})}{(t-q^{2(\lambda_{(r)}-n_{r-1})})(t-\nu^2\, q^{2(n_r-\lambda_{(r)})})} - c(t)\Big) E_{T_{\{\Lambda,M\}}},\end{gathered} \tag{4.5.19}$$

where $k$ is a number of blocks in the diagram $\Lambda$ (4.5.10) and $n_0 = 0$. In the l.h.s. of (4.5.19) we take into account that $y_{M+1} = q^{2(\lambda_{(j)}-n_{j-1})}$, if we add a new cell in the outer corner $(n_{j-1}+1, \lambda_{(j)}+1)|_{j=1,\dots,k+1}$ of $\Lambda$, and $y_{M+1} = \nu^2 q^{2(n_j-\lambda_{(j)})}$, if we remove the cell in the inner corner $(n_j, \lambda_{(j)})|_{j=1,\dots,k}$ of $\Lambda$. Now we compare the residues at $t = q^{2(\lambda_{(j)}-n_{j-1})} =: \mu_j$ and $t = \nu^2 q^{2(n_j-\lambda_{(j)})} =: \nu^2\bar{\mu}_j$ in both sides of Eq. (4.5.19) and deduce

$$\mathrm{Tr}_{D(M+1)}\big(E_{T_{\{\Lambda'_j,M+1\}}}\big) = E_{T_{\{\Lambda,M\}}}\, \frac{1-(\mu_j\bar{\mu}_j)^{-1}}{\lambda} f(\mu_j,q,\nu) \prod_{\substack{r\neq j\\ r=1}}^{k} \frac{\mu_j - \bar{\mu}_r^{-1}}{\mu_j - \mu_r} \; \prod_{r=1}^{k} \frac{\mu_j - \nu^2\mu_r^{-1}}{\mu_j - \nu^2\bar{\mu}_r}, \tag{4.5.20}$$

$$\mathrm{Tr}_{D(M+1)}\big(E_{T_{\{\Lambda''_j,M+1\}}}\big) = E_{T_{\{\Lambda,M\}}}\, \frac{1-(\mu_j\bar{\mu}_j)^{-1}}{\lambda} f(\nu^2\bar{\mu}_j,q,\nu) \prod_{r=1}^{k} \frac{\nu^2\bar{\mu}_j - \bar{\mu}_r^{-1}}{\nu^2\bar{\mu}_j - \mu_r} \prod_{\substack{r\neq j\\ r=1}}^{k} \frac{\bar{\mu}_j - \mu_r^{-1}}{\bar{\mu}_j - \bar{\mu}_r}, \tag{4.5.21}$$

where $f(t,q,\nu) := \frac{(t-\nu^2q^{2n})(t-q^{-1}\nu)(t+q\nu)}{(t-q^{-2n})(t-\nu)(t+\nu)}$. We apply the Ocneanu's (Markov) trace $\mathrm{Tr}_{(1)} \cdots \mathrm{Tr}_{(M)}$ to both sides of Eq. (4.5.20) and find the recurrence relation:

$$\mathrm{qdim}(\Lambda'_j) = \mathrm{qdim}(\Lambda)\, \frac{(1-(\mu_j\bar{\mu}_j)^{-1})}{\lambda} f(\mu_j,q,\nu) \prod_{\substack{r\neq j\\ r=1}}^{k} \frac{\mu_j - \bar{\mu}_r^{-1}}{\mu_j - \mu_r} \; \prod_{r=1}^{k} \frac{\mu_j - \nu^2\mu_r^{-1}}{\mu_j - \nu^2\bar{\mu}_r}, \tag{4.5.22}$$

where the diagram $\Lambda'_j$ is obtained by adding one cell in the outer corner $(n_{j-1}+1, \lambda_{(j)}+1)$ of the diagram $\Lambda$. Note that applying the Ocneanu's (Markov) trace to both sides of the second equation (4.5.21), we deduce the recurrence relation that is equivalent to the relation (4.5.22). It was shown in [246] that the solution of the recurrence relation (4.5.22) is given (up to some factor) by the Wenzl formula [229, 246]:

$$\mathrm{qdim}(\Lambda) = \prod_{(i,j)\in\Lambda} \frac{q^{\frac{1}{2}d_\Lambda(i,j)} - \nu q^{-\frac{1}{2}d_\Lambda(i,j)}}{q^{\frac{1}{2}h_{i,j}} - q^{-\frac{1}{2}h_{i,j}}} \prod_{(i,j)\in\Lambda} \frac{\nu^{-1} q^{\frac{1}{2}d'_\Lambda(i,j)} + q^{-\frac{1}{2}d'_\Lambda(i,j)}}{q^{\frac{1}{2}h_{i,j}} + q^{-\frac{1}{2}h_{i,j}}}, \tag{4.5.23}$$

where $h_{i,j} = (\lambda_i + \lambda^\vee_j - i - j + 1)$ is a hook length (here $\Lambda^\vee = [\lambda^\vee_1, \lambda^\vee_2, \dots]$ is the transpose partition of the partition $\Lambda$) and

$$d_\Lambda(i,j) = \begin{cases} f_{i,j} & \text{if } i \leqslant j \\ f^\vee_{i,j} & \text{if } i > j \end{cases}, \qquad d'_\Lambda(i,j) = \begin{cases} f_{i,j} & \text{if } i < j \\ f^\vee_{i,j} & \text{if } i \geqslant j \end{cases},$$

where $f_{i,j} = \lambda_i + \lambda_j - i - j + 1$ and $f^\vee_{i,j} = -\lambda^\vee_i - \lambda^\vee_j + i + j - 1$.

**Remark 1.** In the $R$-matrix representation of the the $BMW_M(\nu)$ algebras, the generators $\sigma_i$ are given by the $SO_q(N)$, $Sp_q(2n)$ $R$-matrices (3.10.2) (by the $R$-matrices (3.11.52) in the $Osp_q(N, 2m)$ case). For these representations the parameter $\nu$ is fixed (see (3.10.4) and Remark 2 in Subsection **4.4.2**) and we have $\nu = \epsilon q^{\epsilon - N}$, where $\epsilon = +1$ and $\epsilon = -1$ correspond to $SO_q$ and $Sp_q$ cases, respectively. In particular, the formula (4.5.23) is written, for the $SO_q(N)$ $R$-matrix representation of $BMW_M$, in more explicit form (cf. (4.3.92)):

$$\mathrm{qdim}(\Lambda) = \prod_{i=1}^{k} \frac{[N+2(i-1)]_q!}{[\lambda^\vee_i + k - i]_q! [N - \lambda^\vee_i + k - 2 + i]_q!} \cdot \prod_{i<j} [\lambda^\vee_i - \lambda^\vee_j + j - i]_q [N - \lambda^\vee_i - \lambda^\vee_j + i + j - 2]_q, \tag{4.5.24}$$

where $\Lambda^\vee = [\lambda^\vee_1, \lambda^\vee_2, \dots, \lambda^\vee_k] \vdash M$ is the transpose partition of $\Lambda$ and $[h]_q := \frac{q^h - q^{-h}}{q - q^{-1}}$.

**Remark 2.** The analogs of the statements (4.3.79) and (4.3.95) for the Hecke algebras are fairly easy to reformulate and prove for the case of the $BMW_M$ algebras.

## 5. Applications and conclusions

In the previous sections of the paper, we have presented the fundamentals of the theory of quantum groups. We have also considered how to obtain trigonometric and rational (Yangian) solutions of the Yang–Baxter equation on the basis of the theory of quantum Lie groups. Unfortunately, in the previous sections it was not possible for us to discuss in detail the numerous applications of the theory of quantum groups and the Yang–Baxter equation in both theoretical and mathematical physics. In this final section, we shall merely give a brief list of such applications that, in the author's opinion, have some interest.

Before we do this, we recall that in the physics of condensed matter, exactly solvable two-dimensional models are used to describe various layered structures, contact surfaces in electronics, surfaces of superconducting liquids like He II, etc. Two-dimensional integrable field theories are used to describe dynamical effects in one-dimensional spatial systems (such as light tubes, nerve fibers, etc.). In addition, such field theories (and also integrable systems on one-dimensional chains) can also arise on reductions of multidimensional field theories (see, for example, [249]). Quite recently it has been argued that the one-loop dilatation operator (anomalous dimension operator) of the $\mathcal{N} = 4$ Super Yang–Mills theory may be identified, in some restricted cases, with the Hamiltonians of various integrable quantum (super) spin chains [250, 251]. Similar spin chain models (related to the noncompact Lie groups) have previously appeared in the QCD context [252–255].

### 5.1. Quantum periodic spin chains

We have already mentioned that the quantum inverse scattering method [7–9] (an introduction to this method, including the algebraic Bethe ansatz method, that can be readily understood by a wide range of readers, can be found in [256–258]) is designed as a constructive procedure for solving quantum two-dimensional integrable systems. In addition, the quantum inverse scattering method makes it possible to construct quantum integrable systems on one-dimensional chains (see, for example, [103, 162] and [259]). Here we discuss the case of periodic chains. The generalization to the case of open chains will be mentioned in the next Subsection **5.2**. The initial point is the relation (3.9.13) for the $L$-operators, which can be written in the form[27]

$$R_{12}(\theta-\theta')\, L_{K2}(\theta)\, L_{K1}(\theta') = L_{K1}(\theta')\, L_{K2}(\theta)\, R_{12}(\theta-\theta'). \tag{5.1.1}$$

Here $L_{Ki}(\theta)$ are the $N\times N$ matrices in the auxiliary vector space $V_i$, with the matrix coefficients that are the operators in the space of states of the $K$th site of a chain consisting of $M$ sites:

$$L_{Ki}(\theta) = I^{\hat{\otimes}(K-1)}\hat{\otimes} L_i(\theta)\hat{\otimes} I^{\hat{\otimes}(M-K)} \;\to\; [L_{Ki},\, L_{K'i'}] = 0 \;\; (K\neq K'). \tag{5.1.2}$$

In (5.1.2), the symbol $\hat{\otimes}$ denotes a direct product of operator spaces.

To construct an integrable system, we introduce the monodromy matrix

$$T_i(\theta) = D_i^{(1)}\, L_{1i}(\theta)\, D_i^{(2)}\, L_{2i}\dots D_i^{(M)}\, L_{Mi}(\theta). \tag{5.1.3}$$

If the matrices $D^{(K)}$ $(1\leqslant K\leqslant M)$ satisfy the relations

$$R_{ij}(\theta)\, D_j^{(K)}\, D_i^{(K)} = D_j^{(K)}\, D_i^{(K)}\, R_{ij}(\theta), \tag{5.1.4}$$

$$[D_i^{(K)},\, D_j^{(J)}] = [D_i^{(J)},\, L_{Kj}] = 0,$$

then it follows from (5.1.1) that

$$R_{ij}(\theta-\theta')\, T_j(\theta)\, T_i(\theta') = T_i(\theta')\, T_j(\theta) R_{ij}(\theta-\theta'). \tag{5.1.5}$$

The trace of the monodromy matrix (5.1.3) over the auxiliary space $i$ forms the transfer matrix

$$t(\theta) = \mathrm{Tr}_{(i)}\,(T_i(\theta)) \tag{5.1.6}$$

which gives a commuting family of operators: $[t(\theta),\, t(\theta')] = 0$. The commutativity of the transfer matrices follows directly from Eq. (5.1.5) if we multiply it by the matrix $(R_{ij}(\theta-\theta'))^{-1}$ from the right and take the trace $\mathrm{Tr}_{(i,j)}(\dots)$. Using the family of commuting operators $t(\theta)$ a certain local operator $H$ can be constructed, which is interpreted as the Hamiltonian of the system. The locality of the Hamiltonian is a natural physical requirement and means that $H$ describes the interaction of only nearest-neighbor sites of the chain. The remaining operators in the commuting set $t(\theta)$ give an infinite set of integrals of motion indicating the integrability of the constructed system. In many well-known cases, the commuting set is associated with

[27] Usually, this equation is written in the form in which the matrix $R_{12}(\theta)$ is substituted by $R_{21}(\theta)$. This is not important, since $R_{21}(\theta) = R_{12}^{-1}(-\theta)$ satisfies, up to the change of spectral parameters, the same Yang–Baxter equation (3.9.12) as $R_{12}(\theta)$ and all formulas below can be easily adapted to the standard case.

the coefficients in the expansions of $t(\theta)$ over the spectral parameter $\theta$. For example, one can consider logarithmic derivatives of the transfer matrix:

$$\mathcal{I}_n = \frac{d^n}{d\theta^n} \ln \left( t(\theta)\, t(0)^{-1} \right) |_{\theta=0} \tag{5.1.7}$$

and identify the local Hamiltonian with the first logarithmic derivative of the transfer matrix:

$$H \equiv \mathcal{I}_1 = \frac{d}{d\theta} \ln \left( t(\theta)\, t(0)^{-1} \right) |_{\theta=0}, \tag{5.1.8}$$

where the matrix $t(0)^{-1}$ is introduced in order to obtain the local charges $\mathcal{I}_n$ [260].

Now we consider explicit examples of integrable periodic spin chains. It is clear that from the Yang–Baxter equation (3.9.12) there always follow representations for the $L$-operators (5.1.1) in the form of $R$-matrices:

$$\rho_{V_k}\left(L_{Ki}(\theta)\right) = R_{ki}(\theta), \;\; \overline{\rho}_{V_k}\left(L_{Ki}(\theta)\right) = (R_{ik}(\theta))^{-1}. \tag{5.1.9}$$

In this case, the representations of $L_{Ki}(\theta)$ act nontrivially in the space $V_k \otimes V_i$. We choose for $L$-operators the first representation in (5.1.9) and obtain for $T_i(\theta)$ (5.1.3):

$$T_i(\theta) = D_i^{(1)}\, R_{1i}(\theta)\, D_i^{(2)}\, R_{2i}(\theta) \dots D_i^{(M)}\, R_{Mi}(\theta) =$$
$$= \hat{R}'_{i1}(\theta)\, \hat{R}'_{12}(\theta)\, \hat{R}'_{23}(\theta) \dots \hat{R}'_{M-1M}(\theta) P_{M-1\,M} \dots P_{23} P_{12} P_{1i},$$

where $P_{ij}$ are the permutation matrices and $\hat{R}'_{ij}(\theta) = D_i^{(j)}\, \hat{R}_{ij}(\theta)$. Taking the trace $\mathrm{Tr}_{(i)}$, we deduce

$$t(\theta) = \mathrm{Tr}_{(i)} \left( \hat{R}'_{i1}(\theta)\, \hat{R}'_{12}(\theta)\, \hat{R}'_{23}(\theta) \dots \hat{R}'_{M-1M}(\theta) P_{Mi} \right) P_{M-1M} \dots P_{23} P_{12}. \tag{5.1.10}$$

We consider a rather general case of $R$-matrices which can be normalized so that (see, e.g., (3.9.14), (3.9.16), (3.12.21), (3.12.23))

$$\hat{R}_{ij}(\theta) = I + \theta\, h_{ij} + \theta^2 \dots . \tag{5.1.11}$$

These $R$-matrices are called *regular* [154]. For the regular $R$-matrices, using (5.1.10), we obtain

$$t(\theta)\, t(0)^{-1} = I + \theta \left( \sum_{k=1}^{M} h'_{k\,k+1} \right) + \theta^2 \dots , \qquad h'_{k\,k+1} := D_k^{(k+1)}\, h_{k\,k+1} \left( D_k^{(k+1)} \right)^{-1},$$

where $D_M^{(M+1)} := D_M^{(1)}$, $h_{M\,M+1} := h_{M\,1}$ and the local Hamiltonian (5.1.8) is

$$H = \sum_{k=1}^{M} h'_{k\,k+1}. \tag{5.1.12}$$

If we choose the $R$-matrix in (5.1.10) in the form of the trigonometric solution (3.12.21), then we obtain

$$h_{j\,j+1} = \frac{1}{2} \left( \hat{R}_{j\,j+1} + \hat{R}^{-1}_{j\,j+1} - \lambda\, \beta_{\pm}\, \mathbf{K}_{j\,j+1} \right), \tag{5.1.13}$$

where $\beta_\pm = \frac{\alpha_\pm - 1}{\alpha_\pm + 1}$, $\alpha_\pm = \pm q^{\pm 1}\,\nu^{-1}$ and the parameter $\nu$ is fixed for different quantum (super)groups in (3.12.12). We note that the Hamiltonians (5.1.12) with the densities (5.1.13) (and $D^{(k)} = 1$) are the $R$-matrix images of the operators:

$$H_\pm = \frac{1}{2}\sum_{j=1}^{M}\left(\sigma_j + \sigma_j^{-1} + \lambda\,\frac{\nu \mp q^{\pm 1}}{\nu \pm q^{\pm 1}}\,\kappa_j\right),$$

where $\sigma_i$, $\kappa_i$ $(i = 1, \dots, M)$ obey (4.1.1), (4.4.1)–(4.4.3) with periodic identifications: $\sigma_{M+i} = \sigma_i$, $\kappa_{M+i} = \kappa_i$. It is natural to call the algebra with such generators as the periodic Birman–Murakami–Wenzl algebra. The case $\kappa_i = 0$ corresponds to the periodic system with the Hamiltonian:

$$H = \sum_{j=1}^{M}\sigma_j - \frac{\lambda M}{2}, \tag{5.1.14}$$

where $\sigma_i$ are the generators of the periodic $A$-type Hecke algebra $AH_{M+1}$ (see Subsection **4.2**). In the $R$-matrix representation: $\sigma_i \to \hat{R}_i$, $\sigma_M \to \hat{R}_{M\,1}$, where $\hat{R}$ is the $GL_q(2)$ matrix (3.4.8), this Hamiltonian describes the periodic $XXZ$ Heisenberg model.

For the Yangian $R$-matrices (3.12.23) we obtain $SO(N)$ $(\epsilon = +1)$ and $Sp(N)$ $(N = 2n,\ \epsilon = -1)$ invariant spin chain models with local Hamiltonian densities (see, e.g., [103]):

$$h_{l\,l+1} = \left(P_{l\,l+1} + \frac{2}{2\epsilon - N}K^{(0)}_{l\,l+1}\right),$$

where, as usual, $P_{l\,l+1}$ are the transposition matrices, the matrices $K^{(0)}_{l\,l+1}$ were defined in (3.10.9), and for closed chains we imply $O_{M\,M+1} = O_{M\,1}$. The $Osp(N|2m)$ invariant spin chain model corresponds to the densities $h_{l,l+1} = \left(\mathcal{P}_{l\,l+1} + \frac{2}{2+2m-N}\,\mathcal{K}^{(0)}_{l\,l+1}\right)$ which are deduced from (3.12.24). These Yangian models are generalizations of the $XXX$ Heisenberg models of magnets. We recall that the $XXX$ model can be obtained if we take the special limit $q \to 1$ in the $XXZ$ model or choose the $gl(2)$ Yangian $R$-matrix (3.9.16) as a representation of $L$-operators in (5.1.9).

By using (in formulas (5.1.9) and (5.1.10)) the elliptic solution (3.15.3), (3.15.8) of the Yang–Baxter equation, we recover for $N = 2$ the $XYZ$ spin chain model [3, 191], while for $N > 2$ we obtain its integrable generalizations.

At the end of this subsection, we stress that using the transfer matrix (5.1.6), one can construct an integrable 2-dimensional statistical model on the $(M \times L)$ lattice with periodic boundary conditions. Namely, one should consider the partition function

$$Z = \mathrm{Tr}_{(1\dots M)}(\,\underbrace{t(\theta_0)\cdots t(\theta_0)}_{L}\,) = \mathrm{Tr}_{(1\dots M)}\left(\prod_{i=1}^{L}\mathrm{Tr}_{(i)}(D_i^{(1)}L_{1i}(\theta_0)\dots D_i^{(M)}L_{Mi}(\theta_0))\right),$$

where the combination $D_i^{(K)}L_{Ki}(\theta_0)$ (for a special value of the spectral parameter $\theta = \theta_0$) defines the weight of the statistical system in the site $(K, i)$ and $\mathrm{Tr}_{(1\dots M)}$ are the traces over the operator spaces.

### *5.2. Factorizable scattering: S-matrix and boundary K-matrix*

The Yang–Baxter equation (3.9.11):

$$S_{23}(\theta - \theta')\,S_{13}(\theta)\,S_{12}(\theta') = S_{12}(\theta')\,S_{13}(\theta)\,S_{23}(\theta - \theta') \tag{5.2.1}$$

together with the subsidiary relations of unitarity and crossing symmetry

$$S_{12}(\theta)\, S_{21}(-\theta) = I_{12}, \quad S_{12}(\theta) = \big(S_{21}(i\pi - \theta)\big)^{t_1} \tag{5.2.2}$$

uniquely determine factorizable $S$-matrices (with a minimal set of poles) describing the scattering of particle-like excitations in $(1+1)$-dimensional integrable relativistic models [4, 5]. Equations (5.2.2) guarantee that the $S$-matrix $S_{12}(\theta)$ is invertible and skew-invertible (see (3.1.17)). The matrix $S^{i_1 i_2}_{j_1 j_2}(\theta)$ is interpreted as the $S$-matrix for the scattering of two neutral particles with isotopic spins $i_1$ and $i_2$ to two particles with spins $j_1$ and $j_2$, and the spectral parameter $\theta$ is none other than the difference of the rapidities of these particles. For charged particles, the crossing symmetry relation (5.2.2) should be written in the form $S_{12}(\theta) = (S_{2\bar{1}}(i\pi - \theta))^{t_1}$, where the $S$-matrix $S_{2\bar{1}} = S^{i_2 \bar{i}_1}_{k_2 \bar{k}_1}$ describes particle–antiparticle scattering. The many-particle $S$-matrices decompose into products of two-particle matrices (factorization). In this sense, the Yang–Baxter equation (5.2.1) is the condition for the uniqueness of the determination of the many-particle $S$-matrices.

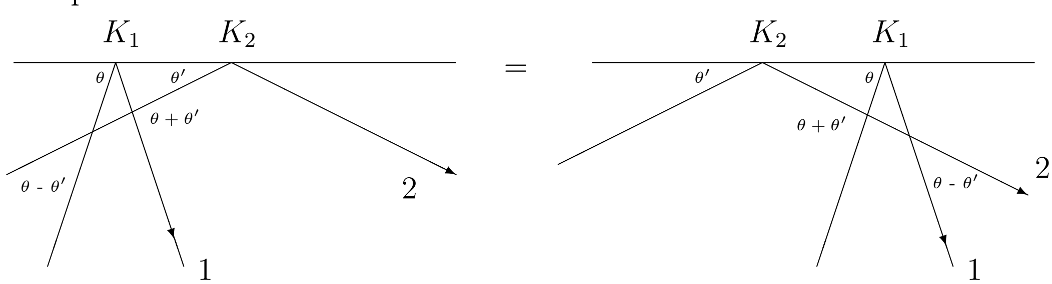
The reflection equation [261–266], which depends on the spectral parameters,

$$S_{12}(\theta - \theta')\, K_1(\theta)\, S_{21}(\theta + \theta')\, K_2(\theta') = K_2(\theta')\, S_{12}(\theta + \theta')\, K_1(\theta)\, S_{21}(\theta - \theta') \tag{5.2.3}$$

determines, together with the unitarity condition

$$K^i_j(\theta)\, K^j_m(-\theta) = \delta^i_m \tag{5.2.4}$$

and relations (5.2.1) and (5.2.2), the factorizable scattering of particles (solitons) on a half-line (see, e.g., [261, 262, 267, 268]). In this case, the operator matrix $K^i_j(\theta)$ describes reflection of a particle with rapidity $\theta$ at a boundary point of the half-line. Graphically, relation (5.2.3) can be represented in the form




We recall [262] that factorizable scattering on a half-line can be described by a Zamolodchikov algebra with generators $\{A^i(\theta)\}$ $(i = 1, \dots, N)$ and boundary operator $B$ that satisfy the defining relations

$$\begin{gathered} A^i(\theta)\, A^j(\theta') = S^{ij}_{kl}(\theta - \theta')\, A^l(\theta')\, A^k(\theta), \quad A^i(\theta)\, B = K^i_j(\theta)\, A^j(-\theta)\, B \;\; \Rightarrow \\ A_{1\rangle}(\theta)\, A_{2\rangle}(\theta') = S_{12}(\theta - \theta')\, A_{2\rangle}(\theta')\, A_{1\rangle}(\theta), \quad A_{1\rangle}(\theta)\, B = K_1(\theta)\, A_{1\rangle}(-\theta)\, B. \end{gathered} \tag{5.2.5}$$

The consistence conditions for this algebra give rise to the Yang–Baxter equation (5.2.1), the unitarity conditions (5.2.2), (5.2.4) and the reflection equation (5.2.3) for the matrices $S$ and $K$.

The reflection equation (5.2.3) can be used [263–266, 269–271] for construction of quantum group invariant integrable spin systems (see, e.g., [12]) on the chains with nonperiodic boundary conditions. Indeed, let $T(\theta)$ be a solution of (5.1.5) (for $R_{ij}(\theta) \equiv S_{ji}(\theta)$) and $K(\theta)$ satisfies (5.2.3). Then the matrix

$$\mathcal{T}(\theta) = T(\theta)\, K(\theta)\, [T(-\theta)]^{-1} \tag{5.2.6}$$

is also a solution of (5.2.3). It can be checked directly, but it also follows from the symmetry transformation

$$A(\theta) \;\to\; [T(\theta)]^{-1} A(\theta), \quad B \;\to\; B,$$

for the algebra (5.2.5) (relations (5.2.5) are obviously invariant under this transformation if we simultaneously substitute $K(\theta) \to T(\theta)K(\theta)[T(-\theta)]^{-1}$).

The matrix $\mathcal{T}(\theta)$ (5.2.6) is called *the Sklyanin monodromy matrix*. By means of this matrix one can construct a partition function for the integrable lattice model with nontrivial boundary conditions defined by the reflection matrix $K(\theta)$. The set of commuting integrals (including the Hamiltonian of the model) is given by the transfer matrix $t(\theta)$ which is constructed as a special trace of $\mathcal{T}(\theta)$:

$$t(\theta) = \mathrm{Tr}\,\big(\mathcal{T}(\theta)\, \overline{K}(\theta)\big) = \mathrm{Tr}\,\big(T(\theta)\, K(\theta)\, [T(-\theta)]^{-1}\, \overline{K}(\theta)\,\big)\,, \tag{5.2.7}$$

where the matrix $\overline{K}(\theta)$ is any solution of the conjugated reflection equation [263–266, 269–271]:

$$S^t_{12}(\theta-\theta')\, \overline{K}{}^t_2(\theta')\, \Psi^t_{12}(\theta+\theta')\, \overline{K}{}^t_1(\theta) = \overline{K}{}^t_1(\theta)\, \Psi^t_{21}(\theta+\theta')\, \overline{K}{}^t_2(\theta')\, S^t_{21}(\theta-\theta'). \tag{5.2.8}$$

Here we require that $\overline{K}(\theta)$ has commutative entries: $[\overline{K}{}^i_j(\theta),\, \overline{K}{}^m_n(\theta')] = 0$,

$$[\overline{K}{}^i_j(\theta),\, K^m_n(\theta')] = 0 = [\overline{K}{}^i_j(\theta),\, T^m_n(\theta')] \;\Rightarrow\; [\overline{K}{}^i_j(\theta),\, \mathcal{T}^m_n(\theta')] = 0.$$

In (5.2.8), we have used the notation $S^t_{12} := S^{t_1 t_2}_{12}$, and the matrix $\Psi_{12}$ is the skew-inverse matrix for $S_{12}$ (cf. (3.1.17)):

$$\Psi^{t_1}_{12}(\theta)\, S^{t_1}_{12}(\theta) = I_{12} = S^{t_1}_{12}(\theta)\, \Psi^{t_1}_{12}(\theta), \quad \Psi_{12}(\theta) = (S^{t_1}_{12}(\theta)^{-1})^{t_1}. \tag{5.2.9}$$

We also assume (see, e.g., [262]) that the matrix $S_{12}(\theta)$ satisfies the cross-unitarity condition (cf. (3.8.9), (3.12.20))

$$S^{t_1}_{12}(\theta)\, \big(D_1^{-1} S_{21}(b-\theta) D_1\big)^{t_1} = \eta(\theta,b) I_{12}, \tag{5.2.10}$$

where $\eta(\theta,b)$ is the scalar function, $b$ is the special parameter which depends on the form of the matrix $S_{12}$, and $D$ is the constant matrix such that: $[D_1 D_2,\, S_{12}(\theta)] = 0$. Comparing Eqs. (5.2.9) and (5.2.10), one can identify

$$\Psi_{12}(\theta) = \frac{1}{\eta(\theta,b)}\, D_1^{-1} S_{21}(b-\theta) D_1,$$

and then rewrite the conjugated reflection equation (5.2.8) in the form

$$S^t_{12}(\theta-\theta')\, \widetilde{K}{}^t_2(\theta')\, S^t_{21}(b-\theta-\theta')\, \widetilde{K}{}^t_1(\theta) = \widetilde{K}{}^t_1(\theta)\, S^t_{12}(b-\theta-\theta')\, \widetilde{K}{}^t_2(\theta')\, S^t_{21}(\theta-\theta'), \tag{5.2.11}$$

where $\widetilde{K}(\theta) = D^{-1}\overline{K}(\theta)$. Note that Eq. (5.2.11) is also one of the consistence conditions but for the "left-boundary" Zamolodchikov algebra (cf. (5.2.5)):

$$\widetilde{B}\, \widetilde{A}_{1\rangle}(\theta) = \widetilde{B}\, \widetilde{K}_1(\theta)\, \widetilde{A}_{1\rangle}(b-\theta), \quad \widetilde{A}_{1\rangle}(\theta)\, \widetilde{A}_{2\rangle}(\theta') = S_{21}(\theta-\theta')\widetilde{A}_{2\rangle}(\theta')\widetilde{A}_{1\rangle}(\theta), \tag{5.2.12}$$

with generators $\widetilde{A}^i$ $(i = 1, \dots, N)$ and left boundary operator $\widetilde{B}$ (we need to consider the condition for the unique reordering of the third-order monomial $\widetilde{B}\,\widetilde{A}_{1\rangle}(\theta)\,\widetilde{A}_{2\rangle}(\theta')$).

The proof of the identity $[t(\theta),\, t(\theta')] = 0$ for the transfer matrix $t(\theta)$ (5.2.7) is straightforward [263] $(\theta^{\pm} = \theta \pm \theta')$:

$$t(\theta')\,t(\theta) = \mathrm{Tr}_{12}\left(\mathcal{T}_2(\theta')\,\mathcal{T}_1^t(\theta)\,\overline{K}_2(\theta')\,\overline{K}_1^t(\theta)\right) =$$

$$= \mathrm{Tr}_{12}\left(\mathcal{T}_2(\theta')\,\mathcal{T}_1^t(\theta)\,S_{12}^{t_1}(\theta^+)\,\Psi_{12}^{t_1}(\theta^+)\,\overline{K}_2(\theta')\,\overline{K}_1^t(\theta)\right) =$$

$$= \mathrm{Tr}_{12}\left(\left(\mathcal{T}_2(\theta')\,S_{12}(\theta^+)\,\mathcal{T}_1(\theta)\right)^{t_1}\,\left(\overline{K}_2^t(\theta')\,\Psi_{12}^t(\theta^+)\,\overline{K}_1^t(\theta)\right)^{t_2}\right) =$$

using Eq. (5.2.3) for $K(\theta) \to \mathcal{T}(\theta)$, we deduce

$$= \mathrm{Tr}_{12}\left(\mathcal{T}_1(\theta)\,S_{21}(\theta^+)\,\mathcal{T}_2(\theta')\,S_{21}^{-1}(\theta^-)\left(\overline{K}_2^t(\theta')\,\Psi_{12}^t(\theta^+)\,\overline{K}_1^t(\theta)\right)^t\,S_{12}(\theta^-)\right) =$$

and applying here the conjugated reflection equation (5.2.8) and transpositions, we finally obtain

$$= \mathrm{Tr}_{12}\left(\left(\mathcal{T}_1(\theta)\,S_{21}(\theta^+)\,\mathcal{T}_2(\theta')\right)^{t_2}\,\left(\overline{K}_1^t(\theta)\,\Psi_{21}^t(\theta^+)\,\overline{K}_2^t(\theta')\right)^{t_1}\right) =$$

$$= \mathrm{Tr}_{12}\left(\mathcal{T}_1(\theta)\,\mathcal{T}_2^t(\theta')\,S_{21}^{t_2}(\theta^+)\,\Psi_{21}^{t_2}(\theta^+)\,\overline{K}_1(\theta)\,\overline{K}_2^t(\theta')\right) = t(\theta)\,t(\theta').$$

Now we take in (5.2.3) the limit $\theta,\, \theta' \to \pm\infty$ in such a way that $\theta - \theta' \to \pm\infty$, and at the same time we set

$$K(\theta)|_{\theta\to\infty} = L, \quad S_{12}(\theta)|_{\theta\to\infty} = R_{12}.$$

$$K(\theta)|_{\theta\to-\infty} = L^{-1}, \quad S_{12}(\theta)|_{\theta\to-\infty} = (R_{21})^{-1}.$$

Then (5.2.3) goes over into (3.2.31), and this is the reason why all algebras with defining relations of type (3.2.31) are called the reflection equation algebras [264–266].

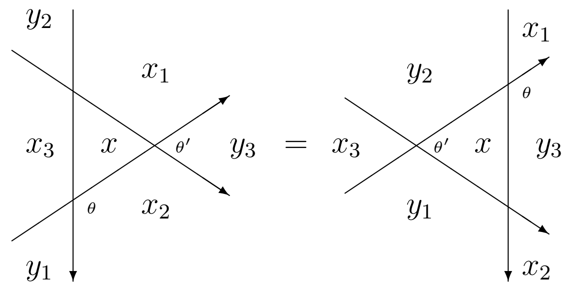
Note that each solution of the Yang–Baxter equation (5.2.1) with the conditions (5.2.2) determines an equivalence class of quantum integrable systems with the given factorizable $S$-matrix. Thus, each classification of solutions to the Yang–Baxter equation is, to some extent, a classification of integrable systems with the properties indicated above.

The 3D analog of the Yang–Baxter (triangle) equation (3.9.15), (5.2.1) is called the *tetrahedron equation* [272, 273] (see also [289]) and defines the consistence condition for 3D factorizable scattering of strings. The 3-dimensional model of such factorizable scattering was first proposed by A. Zamolodchikov in [272, 273]. Then this 3D model was generalized in [274–277]. New solutions of the tetrahedron equation were also considered in [278]. A 3-dimensional version of the 2D reflection equation (5.2.3) (the tetrahedron reflection equation) was proposed in [279]. Combinatorial and algebraic aspects of the 3D reflection equation were considered in [280, 281]. Special solutions of the tetrahedron reflection equation were found in [282].

From a mathematical point of view, higher dimensional generalizations of the Yang–Baxter equations are related to the Manin–Schechtman higher braid groups [283–285], $n$-categories [286–288], and also appeared in the theory of quasitriangular Hopf algebras (see Remark 4 at the end of Subsection **2.5**).

### 5.3. Yang–Baxter equations and calculations of multiloop Feynman diagrams

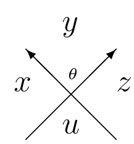
We mention the application of the Yang–Baxter equation in multiloop calculations in quantum field theory. There is a form of the Yang–Baxter equation (see [1, 290] and [289]) that can also be represented in the form of the triangle equation (3.9.15), but the indices $x, x_i, y_i$ are ascribed not to the “lines” but to the “faces”:



(5.3.1)

where $\theta, \theta'$ are angles (spectral parameters), summation is over the index $x$, and

$R^{xy}_{uz}(\theta) \;=$




The analytical form of (5.3.1) is

$$\sum_x R^{x_1\,x}_{y_2\,x_3}(\theta-\theta')\,R^{x\,x_2}_{x_3\,y_1}(\theta)\,R^{x_1 y_3}_{x\,x_2}(\theta') = \sum_x R^{y_2\,x}_{x_3\,y_1}(\theta')\,R^{x_1\,y_3}_{y_2\,x}(\theta)\,R^{y_3\,x_2}_{x\,y_1}(\theta-\theta'). \tag{5.3.2}$$

We have already considered the solution of this equation in Subsection **3.13**. Indeed, one can show that Eq. (3.15.10) is equivalent to Eq. (5.3.2) if we put (for the notation see Subsection **3.13**):

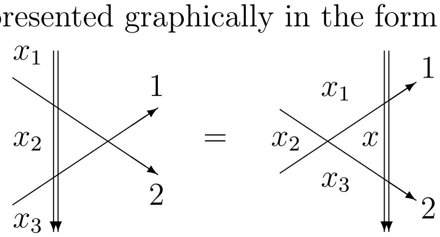
$$R^{x_1\,x}_{x_2\,x_3}(\theta) = \omega^{\frac{1}{2}<x-x_2,\,x_1+x_3>+<x_2,x>}\,W_{x+x_2-x_1-x_3}(\theta), \tag{5.3.3}$$

where the indices $x, x_i$ are 2-dimensional vectors, e.g., $x = (\alpha_1, \alpha_2) \in \mathbf{Z}_N^2$. Thus, (5.3.3), (3.15.8), and (3.15.9) solve the face-type Yang–Baxter equation (5.3.2).

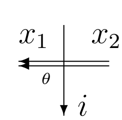
There is a transformation from the vertex-type Yang–Baxter equation (3.9.12) to the face-type (5.3.2) using intertwining vectors $\psi_i^{x_1x_2}$ (see, e.g., [266] and references therein), where $i$ is a vertex index, while $x_1, x_2$ are face indices. The vectors $\psi_i^{x_1x_2}$ satisfy the intertwining relations

$$\psi^{x_2x_1}_{\langle 2}(\theta-\theta')\,\psi^{x_3x_2}_{\langle 1}(\theta)\,R_{12}(\theta') = \sum_x R^{x_1\,x}_{x_2x_3}(\theta')\,\psi^{xx_1}_{\langle 1}(\theta)\,\psi^{x_3x}_{\langle 2}(\theta-\theta') \tag{5.3.4}$$

which are represented graphically in the form (here the angles are the same as in (5.3.1)):



, $\psi_i^{x_1x_2}(\theta) :=$

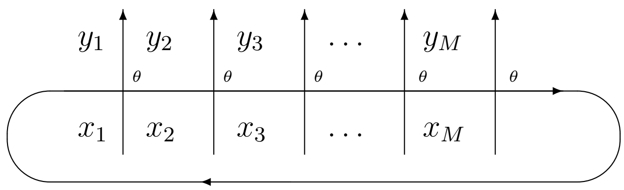



Then the face-type Yang–Baxter equation (5.3.2) is obtained from the vertex equation (3.9.12) if we act on it by $(\psi^{y_2x_1}_{\langle 3}\,\psi^{x_3y_2}_{\langle 2}\,\psi^{y_1x_3}_{\langle 1})$ from the left.

Relations (5.3.1) and (5.3.2), like (3.9.15), give the conditions of integrability of two-dimensional lattice statistical systems (interaction-round face models) with weights determined by the $R$-matrices $R^{xy}_{uz}(\theta)$. In this case, the transfer matrix has the form

$$t^{y_1y_2\ldots y_M}_{x_1x_2\ldots x_M}(\theta) = R^{y_1y_2}_{x_1x_2}(\theta)\,R^{y_2y_3}_{x_2x_3}(\theta)\,R^{y_3y_4}_{x_3x_4}(\theta)\ldots R^{y_My_1}_{x_Mx_1}(\theta),$$

and its graphical representation is

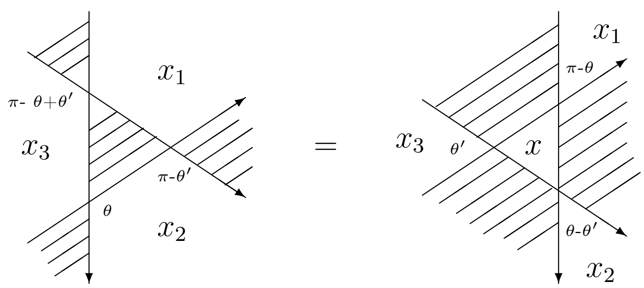



while the partition function for the periodic system on the $(M \times K)$ lattice is given by the standard formula: $Z = \mathrm{Tr}_{1\dots M}(t(\theta))^K$.

We now note that the Yang–Baxter equation (5.3.1), (5.3.2) has a solution in the form $R^{xy}_{uz}(\theta) = G^y_u(\theta)\,\bar{G}^x_z(\pi-\theta)$, where the matrices $G^y_u$, $\bar{G}^x_z = \bar{G}^z_x$ satisfy the star-triangle relation (see, for example, [1] and [290]):

$$f(\theta,\theta')\bar{G}^{x_1}_{x_3}(\pi-\theta+\theta')\,G^{x_2}_{x_3}(\theta)\,\bar{G}^{x_1}_{x_2}(\pi-\theta') = \sum_x G^x_{x_3}(\theta')\,\bar{G}^{x_1}_x(\pi-\theta)\,G^{x_2}_x(\theta-\theta'), \tag{5.3.5}$$

and $f(.,.)$ is an arbitrary function such that $f(\theta,\theta') = f(\theta,\theta-\theta')$. The relations (5.3.5) for $f=1$ can be represented graphically in the form:




The Feynman diagrams, which will be considered here, are graphs with vertices connected by lines labeled by numbers (indices). With each vertex we associate the point in the $D$-dimensional space $R^D$, while the lines of the graph (with index $\alpha$) are associated with the massless Feynman propagator

$$x \;\overset{\alpha}{\rule{3em}{0.4pt}}\; x' \quad = \quad \tfrac{\Gamma(\alpha)}{(x-x')^{2\alpha}}$$

(which is a function of two points $x, x'$ in $D$-dimensional space–time):

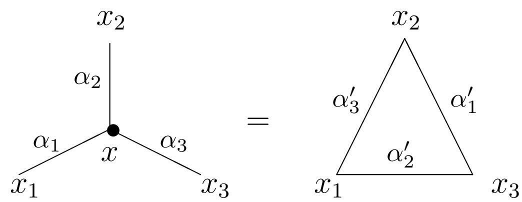
$$G_D(x-x'|\alpha) = \frac{\Gamma(\alpha)}{(x-x')^{2\alpha}} = \frac{\Gamma(\alpha)}{(\sum_\mu (x-x')_\mu (x-x')^\mu)^\alpha}, \tag{5.3.6}$$

where $\Gamma(\alpha)$ is the Euler gamma-function, $D = 4-2\epsilon$ is the dimension of space–time, $(x)_\mu$ $(\mu = 1,2,\dots,D)$ are its coordinates, $\alpha = D/2 - 1 + \eta$, and $\epsilon$ and $\eta$ are, respectively, the parameters of the dimensional and analytic regularizations. The boldface vertices $\bullet$ denote that the corresponding points $x$ are integrated over $R^D$: $\frac{1}{\pi^{D/2}}\int d^D x$. These diagrams are called the Feynman diagrams in the configuration space.

The propagator (5.3.6) satisfies the relation

$$\int \tfrac{d^D x}{\pi^{D/2}} \prod_{i=1}^{3} G_D(x-x_i|\alpha_i) \overset{\sum \alpha_i = D}{=} G_D(x_1-x_2|\alpha'_3)\,G_D(x_2-x_3|\alpha'_1)\,G_D(x_3-x_1|\alpha'_2), \tag{5.3.7}$$

which is represented as the star-triangle identity for the Feynman diagrams:




where $\alpha_1+\alpha_2+\alpha_3 = D$ and $\alpha'_i := D/2-\alpha_i$. Equation (5.3.7) can be readily derived if we put $(x_3)^\mu = 0\;\forall\mu$ and make in the right-hand side of (5.3.7) a simultaneous inversion transformation of the variables of integration, $(x)_\mu \to (x)_\mu/x^2$ and the coordinates $(x_{1,2})^\mu$. Relations (5.3.5) and (5.3.7) are equivalent if we set

$$G^x_{x'}(\theta) = \bar{G}^x_{x'}(\theta) = G_D(x-x'|\frac{D}{2}(1-\frac{\theta}{\pi})),\quad f(\theta,\theta')=1,\quad \sum_x = \int \frac{d^Dx}{\pi^{D/2}}. \tag{5.3.8}$$

Thus, the analytically and dimensionally regularized massless propagator (5.3.6) satisfies the infinite-dimensional star-triangle relation (5.3.5) and accordingly, on the basis of (5.3.6) and (5.3.8), we can construct solutions of the Yang–Baxter equation (5.3.1), (5.3.2). This remark was made in [290], in which calculations were carried out of vacuum diagrams with an infinite number (in the thermodynamical limit) of vertices corresponding to a planar square lattice ($\phi^4$ theory, $D=4$), a planar triangular lattice ($\phi^6$ theory, $D=3$), and a honeycomb lattice ($\phi^3$ theory, $D=6$). The star-triangle relation (5.3.7) (known also as *the uniqueness relation*) was used in addition for analytic calculation of the diagrams that contribute to the 5-loop $\beta$-function of the $\phi^4_{D=4}$ theory [291] and of massless ladder diagrams [292–294, 297]. By means of identity (5.3.7) the symmetry groups of dimensionally and analytically regularized massless diagrams were investigated [295, 297], [302][28]. We emphasize that an extremely interesting problem is that of massive deformation of the propagator function (5.3.6) and the corresponding deformation of the star-triangle relation (5.3.7).

There is an elegant operator interpretation [297, 298] of the star-triangle identity (5.3.7). Indeed, consider the $D$-dimensional Heisenberg algebra $\mathcal{H}_D$ as the algebra of functions of the generators $\hat{q}_\mu = \hat{q}^\dagger_\mu$ and $\hat{p}_\mu = \hat{p}^\dagger_\mu$ ($\mu = 1,\dots,D$) subject to the defining relations

$$[\hat{q}_\mu,\,\hat{p}_\nu] = \mathbf{i}\,\delta_{\mu\nu},\quad (\mu,\nu = 1,2,\dots,D), \tag{5.3.9}$$

where $\hat{q}_\mu$ and $\hat{p}_\mu$ are the operators of the coordinate and momentum, respectively. Consider a representation of the algebra (5.3.9) in the linear vector space of complex functions $\psi(x) := \psi(x_\mu)$ on $\mathbb{R}^D$:

$$\hat{q}_\mu\,\psi(x) = x_\mu\,\psi(x),\quad \hat{p}_\mu\,\psi(x) = -\mathbf{i}\,\partial_\mu\,\psi(x).$$

It is convenient to realize the action of elements $\hat{A}\in\mathcal{H}_D$ as the action of integral operators: $\hat{A}\,\psi(x) = \int d^Dy\,\langle x|\hat{A}|y\rangle\,\psi(y)$. The integral kernels $\langle x|\hat{A}|y\rangle$ can be considered as matrix elements of $\hat{A}$ for the states $|x\rangle := |\{x_\mu\}\rangle$ and $\langle y| = |y\rangle^\dagger$ such that

$$\langle y|x\rangle = \delta^D(y-x),\quad \hat{q}_\mu|x\rangle = x_\mu\,|x\rangle,\quad \int d^Dx\,|x\rangle\,\langle x| = \hat{1}. \tag{5.3.10}$$

We extend the algebra $\mathcal{H}_D$ by the elements $\hat{q}^{2\alpha} := (\hat{q}^\mu\hat{q}_\mu)^\alpha$ and pseudo-differential operators $\hat{p}^{-2\beta} := (\hat{p}^\mu\hat{p}_\mu)^{-\beta}$ ($\forall\alpha,\beta\in\mathbb{C}$). The corresponding integral kernels are

$$\langle x|\hat{q}^{2\alpha}|y\rangle = x^{2\alpha}\,\delta^D(x-y),\quad \langle x|\frac{1}{\hat{p}^{2\beta}}|y\rangle = a(\beta)\,\frac{1}{(x-y)^{2\beta'}}, \tag{5.3.11}$$

[28] Here the symmetry of diagrams means the symmetry of the corresponding perturbative integrals.

where $a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2}\, 2^{2\beta}\, \Gamma(\beta)}$, $\beta' = D/2 - \beta$ and $\beta' \neq 0, -1, -2, \dots$.

For the extended Heisenberg algebra one can prove [297, 298] that the operators $H_\alpha := \hat{p}^{2\alpha}\, \hat{q}^{2\alpha}$ ($\forall \alpha \in \mathbb{C}$) form a commutative family. The commutativity condition $[H_\alpha,\, H_{-\beta}] = 0$ is represented in the form

$$\hat{p}^{2\alpha}\hat{q}^{2\gamma}\hat{p}^{2\beta} = \hat{q}^{2\beta}\hat{p}^{2\gamma}\hat{q}^{2\alpha}, \quad (\gamma = \alpha + \beta). \tag{5.3.12}$$

Then it is not hard to see that this identity, written for integral kernels by means of (5.3.11), is equivalent to the star-triangle relation (5.3.7). One should act on (5.3.12) by vectors $\langle x_1 - x_3|$ and $|x_2 - x_3\rangle$ from the left and right, respectively, and insert, in the l.h.s. of (5.3.12), the unit $\hat{1}$ (5.3.10):

$$\langle x_1 - x_3|\hat{p}^{2\alpha} \left(\int d^D x\, |x\rangle\, \langle x|\right) \hat{q}^{2\gamma}\hat{p}^{2\beta}|x_2 - x_3\rangle = \langle x_1 - x_3|\hat{q}^{2\beta}\hat{p}^{2\gamma}\hat{q}^{2\alpha}|x_2 - x_3\rangle \;\;\Rightarrow$$

$$\int d^D x\, \langle x_1 - x_3|\hat{p}^{2\alpha}|x\rangle\, x^{2\gamma}\, \langle x|\hat{p}^{2\beta}|x_2 - x_3\rangle = (x_1 - x_3)^{2\beta}\langle x_1 - x_3|\hat{p}^{2\gamma}|x_2 - x_3\rangle(x_2 - x_3)^{2\alpha}.$$

Applying here the second equation in (5.3.11), we obtain (5.3.7) for $\alpha = -\alpha_1'$, $\beta = -\alpha_2'$ and $\gamma = -\alpha_3$.

Consider the set of Heisenberg algebras $\mathcal{H}_D$ with the generators $\{\hat{q}^\mu_{(a)}, \hat{p}^\nu_{(b)}\}$ $(a, b = 1, 2, \dots, N)$ such that: $[\hat{q}^\mu_{(a)},\, \hat{p}^\nu_{(b)}] = \mathrm{i}\, \delta^{\mu\nu}\, \delta_{ab}$. Then the star-triangle identity (5.3.12) is obviously generalized as

$$(\hat{q}_{(ab)})^{2\alpha}(\hat{p}_{(b)})^{2(\alpha+\beta)}(\hat{q}_{(ab)})^{2\beta} = (\hat{p}_{(b)})^{2\beta}(\hat{q}_{(ab)})^{2(\alpha+\beta)}(\hat{p}_{(b)})^{2\alpha}, \tag{5.3.13}$$

where $\hat{q}^\mu_{(ab)} = \hat{q}^\mu_{(a)} - \hat{q}^\mu_{(b)}$. Taking into account (5.3.13), one can directly check that for an arbitrary parameter $\xi$ the operator

$$R_{ab}(\alpha; \xi) := (\hat{q}_{(ab)})^{2(\alpha+\xi)}(\hat{p}_{(a)})^{2\alpha}(\hat{p}_{(b)})^{2\alpha}(\hat{q}_{(ab)})^{2(\alpha-\xi)} = 1 + \alpha\, h_{(ab)}(\xi) + \alpha^2 \dots \tag{5.3.14}$$

is a regular (see (5.1.11)) solution of the Yang–Baxter equation:

$$R_{ab}(\alpha; \xi)\, R_{bc}(\alpha + \beta; \xi)\, R_{ab}(\beta; \xi) = R_{bc}(\beta; \xi)\, R_{ab}(\alpha + \beta; \xi)\, R_{bc}(\alpha; \xi). \tag{5.3.15}$$

The solution (5.3.14) for arbitrary $D$ and $\xi = 1$ was found in [298] and for any $\xi$ in [299]. The factorized form of the solution (5.3.14) (for $D = 1$) reminds the factorization of $R$-matrices observed in [300, 301].

Using the standard procedure (see Eqs. (5.1.11), (5.1.12)), one can construct an integrable system with a Hamiltonian that is related to the $R$-matrix (5.3.14):

$$H(\xi) = \sum_{a=1}^{N-1} h_{(a,a+1)}(\xi), \tag{5.3.16}$$

where the Hamiltonian densities $h_{(ab)}(x)$ are derived from (5.3.14)

$$\begin{aligned} h_{(ab)}(\xi) &= 2\, \ln(\hat{q}_{(ab)})^2 + (\hat{q}_{(ab)})^{2\xi}\, \ln(\hat{p}^{\,2}_{(a)}\, \hat{p}^{\,2}_{(b)})\, (\hat{q}_{(ab)})^{-2\xi} = \\ &= \hat{p}^{-2\xi}_{(a)}\, \ln(\hat{q}_{(ab)})^2\, \hat{p}^{\,2\xi}_{(a)} + \hat{p}^{-2\xi}_{(b)}\, \ln(\hat{q}_{(ab)})^2\, \hat{p}^{\,2\xi}_{(b)} + \ln(\hat{p}^{\,2}_{(a)}\, \hat{p}^{\,2}_{(b)}). \end{aligned} \tag{5.3.17}$$

For $D = 1$ and $\xi = 1/2$ the Hamiltonian (5.3.16) reproduces the Hamiltonian for the Lipatov integrable model [252, 253].

A remarkable fact is that for the algebra with the generators $\{\hat{q}^\mu_{(a)}, \hat{p}^\nu_{(b)}\}$ one can define a trace. In particular, we need to define correctly the $D$-dimensional integral:

$$\int d^D\! x\, \langle x|\hat{q}^{2\alpha_1}\hat{p}^{2\beta_1}\hat{q}^{2\alpha_2}\hat{p}^{2\beta_2}\dots\hat{q}^{2\alpha_n}\hat{p}^{2\beta_n}|x\rangle = c(\alpha_i, \beta_j)\int \frac{d^D\! x}{x^{2\gamma}} \equiv \mathrm{Tr}(\hat{q}^{2\alpha_1}\hat{p}^{2\beta_1}\dots\hat{q}^{2\alpha_n}\hat{p}^{2\beta_n}), \tag{5.3.18}$$

where $\gamma = D/2 + \sum_i(\beta_i - \alpha_i)$ and $c(\alpha_i, \beta_j)$ is the coefficient function. Recall that the dimension regularization scheme requires the identity [303]:

$$\int \frac{d^D x}{x^{2(D/2+\alpha)}} = 0 \quad \forall \alpha \neq 0, \tag{5.3.19}$$

and the integral (5.3.18) looks meaningless. However, we can extend the definition for (5.3.19) at the point $\alpha = 0$ and, thus, define the formal expression (5.3.18). The definition is [295]:

$$\int \frac{d^D x}{x^{2(D/2+\alpha)}} = \pi \Omega_D \delta(|\alpha|), \tag{5.3.20}$$

where $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the area of the unit hypersphere in $\mathbb{R}^D$, $\alpha = |\alpha| e^{i \arg(\alpha)}$ and $\delta(.)$ is the one-dimensional delta-function. The cyclic property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for the trace (5.3.18) can be checked directly. The trace operation (5.3.18) permits one to reduce [295] the evaluation of propagator-type perturbative integrals (and searching for their symmetries) to the evaluation of vacuum perturbative integrals. Further, breaking any of the propagators in the vacuum diagram, one can obtain many remarkable nontrivial relations between the propagator-type $D$-dimensional integrals. Sometimes these relations are called "glue-and-cut" symmetry (for details see [295, 296])).

One can deduce another star-triangle relation [304] ($x_i \in \mathbb{R}^D$):

$$\left(\frac{2\bar{\alpha}_1 \bar{\alpha}_3}{\alpha_2}\right)^{D/2} W(x_3 - x_1|\alpha_1)\, W(x_1 - x_2|\alpha_2)\, W(x_2 - x_3|\alpha_3) =$$

$$= \int \frac{d^D x}{\pi^{D/2}}\, W(x_1 - x|\bar{\alpha}_3)\, W(x_3 - x|\bar{\alpha}_2)\, W(x_2 - x|\bar{\alpha}_1), \tag{5.3.21}$$

where $W(x|\alpha) = \exp\left(-x^2/(2\alpha)\right)$ and the map

$$\bar{\alpha}_i = \frac{\alpha_1 \alpha_2 \alpha_3}{(\alpha_1 + \alpha_2 + \alpha_3)} \frac{1}{\alpha_i}, \quad \alpha_i = \frac{\bar{\alpha}_1 \bar{\alpha}_3 + \bar{\alpha}_2 \bar{\alpha}_3 + \bar{\alpha}_1 \bar{\alpha}_2}{\bar{\alpha}_i}, \tag{5.3.22}$$

is the well known star-triangle transformation for resistances in electric networks. The identity (5.3.21) is related to the local Yang–Baxter equation [305] and is also rewritten in the operator form [304]

$$W(\hat{q}\,|\,\alpha_1)\, W(\hat{p}\,|\,\alpha_2^{-1})\, W(\hat{q}\,|\,\alpha_3) = W(\hat{p}\,|\,\bar{\alpha}_3^{-1})\, W(\hat{q}\,|\,\bar{\alpha}_2)\, W(\hat{p}\,|\,\bar{\alpha}_1^{-1}). \tag{5.3.23}$$

To obtain (5.3.21) from (5.3.23), we have used the representations

$$\langle x| e^{\frac{1}{\alpha}\hat{q}^2} |y\rangle = e^{\frac{1}{\alpha}(x)^2}\, \delta^D(x-y), \quad \langle x| e^{-\frac{1}{2}\alpha \hat{p}^2} |y\rangle = (2\pi\alpha)^{-D/2}\, e^{-\frac{1}{2\alpha}(x-y)^2}.$$

It is tempting to apply identities (5.3.21)–(5.3.23) for investigation of symmetries and analytical calculations of massive perturbative multiloop integrals written in the $\alpha$-representation. Besides, we hope that the local star-triangle relations (5.3.21), (5.3.23) will help in constructing a massive deformation of the star-triangle relations (5.3.7), (5.3.12). The generalizations of the star-triangle relations (5.3.7), (5.3.12) for spinorial and tensor particles were considered in [297, 306–308].

**Remark 1.** It is also necessary to note the possible applications of the above methods to the calculations of planar multiloop Feynman integrals arising in the fishnet conformal field theories [309] (in particular, see [310–316] and references therein).

**Remark 2.** Note that we have not considered at all the numerous applications of quantum Lie groups and algebras with deformation parameters $q$ satisfying the conditions $q^N = 1$, i.e., when the parameters $q$ are equal to the roots of unity. These applications (see, for example, [317–326] and references therein) appear mostly in the context of the topological and 2D conformal field theories and are associated with the specific theory of representations of such quantum groups that, generally speaking, can no longer be regarded as the deformation of the classical Lie groups and algebras.

**Acknowledgements**

We would like to thank G. Arutyunov, L. Castellani, S. Derkachov, A. N. Kirillov, P. P. Kulish, A. I. Molev, O. V. Ogievetsky, Z. Popowicz, P. N. Pyatov, and V. O. Tarasov for valuable discussions and comments. We are especially grateful to O. V. Ogievetsky for discussing the content of Section 4.

**Conflict of Interest**

The author declares no conflict of interest.

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